Chapter #5

TRANSIENT ANALYSIS USING STATE VARIABLES

5.1 INTRODUCTION

When the dynamic behavior of a circuit is under consideration, the equations representing the circuit, say in node or mesh analysis, are generally integro-differential. They can then be transformed into one scalar differential equation of the second or higher order. However, the differential equations of a circuit may also be written as a set of first-order differential equations, or when expressed in matrix form it results in a first-order vector differential equation of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{w}, t),$$

where $\mathbf{x}$ is a vector of unknown variables called *state variables*, $\mathbf{w}$ represents the set of inputs and t is the time.

The set of first-order differential equations written in such a form is called a *state equation* and the vector $\mathbf{x}$ represents the *state* of the network. State equations play an important role in the study of the dynamic behavior of a circuit. There are three basic advantages in using the state equations in this form. (1) There is an enormous amount of mathematical knowledge for solving such equations while the equations by themselves can be derived from formal topological properties of the circuit, using the matrix approach. (2) It can be easily and naturally extended to nonlinear and time-varying or switched networks and is, in fact, the approach most often used in characterizing such networks and (3) it is easily programmed for and solved by computers.

In this chapter, we shall formulate, derive and solve first-order vector differential equations, i.e. state equations. As before, we shall be limited here to linear, time-invariant circuits that may be reciprocal or nonreciprocal. On the other hand, this approach is applicable to circuits of any complicity, especially with computer-aided analysis. In this study, when using a computer is suggested, we are referring to the MATHCAD or MATHLAB programs which are also suitable for symbolic computation.

5.2 THE CONCEPT OF STATE VARIABLES

Two general methods of circuit analysis are usually studied in-depth in introductory courses in circuit analysis(*), namely nodal analysis and mesh analysis. Both of these methods are very useful for resistive d.c. and *RLC* a.c. circuits in their steady-state behavior. The basic variables in these two kinds of circuits, node voltages and mesh currents, were constant quantities, i.e. with no variation in time. Thus, the nodal and mesh equations in such circuits happen to be algebraic equations, without derivatives and integrals. However, node voltages or mesh currents when used as basic variables in *transient analysis* are expressed as a function of time. Therefore, the node and loop equations here are in general integro-differential equations of the second order.

Consider, as an example, the circuit in Fig. 5.1, in which the inductor current and two capacitor currents may be expressed as

$$i_{L2} = \frac{1}{L_2}\int_0^t (v_{n1} - v_{n2})d\tau + I_0, \tag{5.1a}$$

$$\left.\begin{aligned} i_4 &= C_4\frac{dv_{C4}}{dt} = C_4\frac{dv_{n2}}{dt} \\ i_5 &= C_5\frac{dv_{C5}}{dt} = C_5\frac{dv_{n3}}{dt} \end{aligned}\right. \tag{5.1b}$$

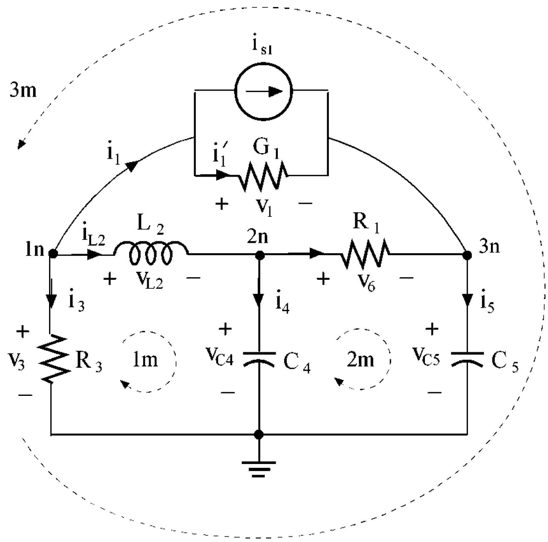

Figure 5.1 Circuit of the example for writing node and mesh equations.

(*)See for example W. H. Hayt and J. E. Kemmerly (1998) *Engineering Circuit Analysis*, McGraw-Hill.

Then the node equations may be written by inspection of the circuit as:

$$(G_1+G_2)v_{n1}+\frac{1}{L_2}\int_0^t v_{n1}d\tau-\frac{1}{L_2}\int_0^t v_{n2}d\tau-G_1v_{n3}=-i_{s1}-I_0$$

$$-\frac{1}{L_2}\int_0^t v_{n1}d\tau+G_6v_{n2}+C_4\frac{dv_{n2}}{dt}+\frac{1}{L_2}\int_0^t v_{n2}d\tau-G_6v_{n3}=I_0$$

$$-G_1v_{n1}-G_6v_{n3}+C_5\frac{dv_{n3}}{dt}=i_{s1}. \qquad (5.2)$$

Once these equations are solved for the node voltages v_{n1}, v_{n2} and v_{n3}, the remaining variables are easily obtained.

However, the presence of the integrals of unknowns in node equations 5.2 causes some difficulties in the solution. The integrals can be eliminated by differentiating the equations in which they appear, but this will increase the order of the derivatives. An easier way of analyzing would be if we avoid the appearance of the integrals altogether. We note that an integral appears in the present example of node equations when the current of an inductor is eliminated by using equation 5.1a. In a similar way, the integrals appear in mesh equations when the voltages of the capacitors are eliminated by substituting their v–i relationship. Therefore these integrals will not appear if we leave both the capacitor voltages and inductor currents as variables using a mixed set of equations, i.e. based on Kirchhoff's laws.

Let us illustrate this idea of using capacitor voltages and inductor currents as unknown variables in the same example of the circuit in Fig. 5.1. We may write three independent KCL equations for the nodes $1n$, $2n$ and $3n$, and three KVL equations for loops (meshes) indicated by the dashed arrows:

$$\begin{aligned} i'_1+i_{L2}+i_3&=-i_{s1},\\ -i_{L2}+i_4+i_6&=0, \qquad (5.3a)\\ -i'_1+i_5-i_6&=i_{s1},\end{aligned}$$

$$\begin{aligned} v_{L2}+v_{C4}-v_3&=0,\\ -v_{C4}+v_6+v_{C5}&=0, \qquad (5.5b)\\ v_3-v_{C5}-v_1&=0.\end{aligned}$$

Substituting equation 5.1b for i_4 and i_5, taking into consideration that $L_2(di_{L2}/dt)=v_{L2}$ and eliminating all branch voltages except for the capacitor voltages by using the v–i relationships, and after rearranging the terms, yields

$$\begin{aligned} C_4\frac{dv_{C4}}{dt}&=i_{L2}=i_6,\\ C_5\frac{dv_{C5}}{dt}&=i'_1+i_6+i_{s1}, \qquad (5.4)\\ L_2\frac{di_{L2}}{dt}&=-v_{C4}+R_3i_3\end{aligned}$$

$$R_6 i_6 = v_{C5} - v_{C4} \tag{5.5a}$$

$$\begin{aligned} i_1' + i_3 &= i_{L2} - i_{s1} \\ R_1 i_1' - R_3 i_3 &= v_{C5}. \end{aligned} \tag{5.b}$$

These are six equations in six unknowns. However, we can reduce the number of equations that must be solved simultaneously. We note that equations 5.5a and 5.5b are algebraic, i.e., they contain no derivatives or integrals. They can be used to eliminate the rest of the unknown variables in (5.4) except v_{C4}, v_{C5} and i_{L2}, whose derivatives are involved in these equations. The algebraic equations 5.5a and 5.5b can be easily solved (the first one trivially) to yield

$$\begin{aligned} i_6 &= -\frac{1}{R_6} v_{C4} + \frac{1}{R_6} v_{C5} \\ i_1' &= \frac{1}{R_1 + R_3} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} - \frac{R_3}{R_1 + R_2} i_{s1} \\ i_3 &= -\frac{1}{R_1 + R_3} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} - \frac{R_1}{R_1 + R_3} i_{s1}. \end{aligned} \tag{5.6}$$

Finally, these equations can be substituted into equation 5.4 to yield, after rearrangement,

$$\begin{aligned} C_4 \frac{dv_{C4}}{dt} &= \frac{1}{R_6} v_{C4} - \frac{1}{R_6} v_{C5} + i_{L2} \\ C_5 \frac{dv_{C5}}{dt} &= -\frac{1}{R_6} v_{C4} - \frac{R_1 + R_3 + R_6}{R_6(R_1 + R_3)} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} + i_{s1} \\ L_2 \frac{di_{L2}}{dt} &= -v_{C4} - \frac{R_3}{R_1 + R_3} v_{C5} + \frac{R_3 R_1}{R_1 + R_3} i_{L2} - \frac{R_1 R_3}{R_1 + R_3} i_{s1}, \end{aligned} \tag{5.7a}$$

or in matrix form, after dividing by the coefficients on the left,

$$\frac{d}{dt}\begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix} = \begin{bmatrix} \frac{1}{C_4 R_6} & -\frac{1}{C_4 R_6} & \frac{1}{C_4} \\ -\frac{1}{C_5 R_6} & \frac{R_1 + R_3 + R_6}{C_5 R_6 (R_1 + R_3)} & \frac{R_3}{C_5 (R_1 + R_3)} \\ -\frac{1}{L_2} & -\frac{R_3}{L_2 (R_1 + R_3)} & \frac{R_1 R_3}{L_2 (R_1 + R_3)} \end{bmatrix} \begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \frac{1}{C_5} \\ -\frac{R_1 R_3}{L_2 (R_1 + R_3)} \end{bmatrix} i_{s1}. \tag{5.7b}$$

The resulting *matrix equation* 5.7b represents three first-order differential equations in three unknowns. It is called the **state equation** and the variables v_{C4}, v_{C5} and i_{L2} are called the **state variables**.

As can be seen, the advantage of this method is that no integrals appear, and subsequently no second derivatives occur as a result of the differentiation. The initial conditions, or **initial state** of the circuit, are the initial values of the capacitor voltages and inductor currents, which usually can be independently specified in the circuit, i.e. their values just after t_0 are determined by their values just before t_0. This is the second reason for choosing capacitor voltages and inductor currents as unknown variables.

Further advantages in describing the network by first-order differential equations are:

1) A simple systematic method for writing such equations can be formulated by using the graph theory.
2) A systematic matrix solution may be applied for solving these first-order differential equations. It may be easily programmed for a numerical and symbolic solution with appropriate computer software.
3) It is quite easy to extend the state-variable representation to time-varying and nonlinear networks.

The concept of *state variables*, or just **state**, satisfies two basic conditions of circuit analysis:

a) If at any time, say t_0, the state is known (which is the initial condition or initial state), then the state equations uniquely determine the state at any time $t > t_0$ for any given input. In other words, given the state of the circuit at time t_0 and all the inputs, the behavior of the circuit is completely determined for all $t > t_0$.

b) The state and the input uniquely determine the value of the remaining circuit variables.

Proof a) From the theory of differential equations we know that the initial values of the variables uniquely define, by differential equations, such as 5.7, the value of the variables for all $t \geq t_0$. In other words, the state $(v_C(t), i_L(t))$ can be expressed by the state equations in terms of the initial state.

Proof b) We may use the substitution (or compensation) principle, which states that in any linear circuit any voltage drop across a passive element, say the capacitance, may be substituted by an independent voltage source equal to this drop. In addition, any current through a passive element, say the inductance, may be substituted by an independent current source equal to this current. Hence, we will replace all the inductors by independent current sources whose values $i_L(t)$ are given by the found state variables and all the capacitors by independent voltage sources whose values are equal to the found state variables $v_C(t)$. As a result, we will obtain a pure resistive network in which any variable can be determined by any well-known method of resistive circuit analysis.

For example, let the desired output quantities be v_3 and v_6 in the circuit being considered in Fig. 5.1. Since $v_3 = R_3 i_3$ and $v_6 = R_6 i_6$, by multiplying the third and the first equations of 5.6 correspondingly by R_3 and R_6, we have

$$v_3 = -\frac{R_3}{R_1 + R_3} v_{C5} + \frac{R_1 R_3}{R_1 + R_3} i_{L2} - \frac{R_1 R_3}{R_1 + R_3} i_{s1}$$

$$v_6 = -v_{C4} + v_{C5},$$

where v_{C4}, v_{C5} and i_{L2} represent the voltage and current sources, which substitute the elements C_4, C_5 and L_2 subsequently. The above expressions in matrix form are

$$\begin{bmatrix} v_3 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{R_3}{R_1 + R_3} & \dfrac{R_1 R_3}{R_1 + R_3} \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix} + \begin{bmatrix} -\dfrac{R_1 R_3}{R_1 + R_3} \\ 0 \end{bmatrix} [i_{s1}]. \quad (5.8)$$

This matrix equation is called an output equation.

Both the state equation 5.7b and the output equation 5.8 equations may be written in compact matrix notation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{w} \quad (5.9a)$$

$$\mathbf{y} = \mathbf{c}\mathbf{x} + \mathbf{d}\mathbf{w}, \quad (5.9b)$$

where $\mathbf{x}$ is the state vector, $\mathbf{w}$ is the input and $\mathbf{y}$ is the output vector. The meanings of matrixes, $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$, which are dependent upon circuit elements, are obvious from equations 5.7b and 5.8.

Next, we shall consider the number of independent state variables that represent the transient behavior of a network.

5.3 ORDER OF COMPLEXITY OF A NETWORK

As is known, node-voltage, mesh-currents, and mixed variable equations (based on Kirchhoff's two laws) completely represent any electrical circuit. Recall that the number of independent node-voltage equations, i.e., number of independent Kirchhoff's current law (KCL) equations, is $B - (N - 1)$, where B is the number of branches and N is the number of nodes. These numbers are determined only by the graph of the circuit and not by the types of the branches, i.e. they would not be influenced if the branches were all resistors, or if some were capacitors and/or inductors. However, in resistive circuits driven by d.c. sources the node or mesh equations are algebraic, with no variation in time. On the other hand, when capacitors or inductors are present, the equations will be integro-differential. Hence, the question is how many independent variables represent the circuit in its transient (dynamic) behavior. We know that each capacitor and each inductor introduces a variable in such behavior since the v-i characteristic of each contains a derivative or integral. We also know that, for a unique solution of differential equations, the arbitrary constants have to be determined.

The number of these constants is equal to the number of independent initial conditions that can be specified in a circuit. It is also known that the number of initial conditions is related to the energy-storing elements, capacitors and inductors, and in general is equal to the number of such elements in the circuit. The exceptions are the, so-called, **all-capacitor loops** and **all-inductor cut-sets**. Consider the circuit shown in Fig. 5.2. There are five energy-storing elements, but in this circuit there is an all-capacitor loop, consisting of two capacitors C_1 and C_2 and a voltage source, and an all-inductor cut-set (see dashed line in Fig. 5.2) consisting of three inductors L_3, L_4 and L_5. In this case, the capacitor voltages and inductor currents will be restricted by KVL and KCL, namely

$$v_{C1} + v_{C2} = v_{s8} \tag{5.10a}$$

$$i_{L4} + i_{L5} = i_{L3}, \tag{5.10b}$$

which means that one of the voltages and one of the currents can be determined if the other is known. This also means that the initial values of both v_{C1} and v_{C2} cannot be prescribed independently, nor can the initial values of all three currents i_{L3}, i_{L4} and i_{L5}. Therefore, each of the constraint relationships, such as equations 5.10a and 5.10b, reduce the number of independent variables.

In other words, the order of complexity of any network equals the total number of *energy-storing elements minus the number of all-capacitor loops and the number of all-inductor cut-sets*. Thus, the order of complexity of the circuit of Fig. 5.2 is $5 - 1 - 1 = 3$. Note that (1) all-capacitor loops may also consist of ideal voltage sources and all-inductor cut-sets may also include ideal current sources, and (2) only independent all-capacitor loops and all-inductor cut-sets are taken into account(*).

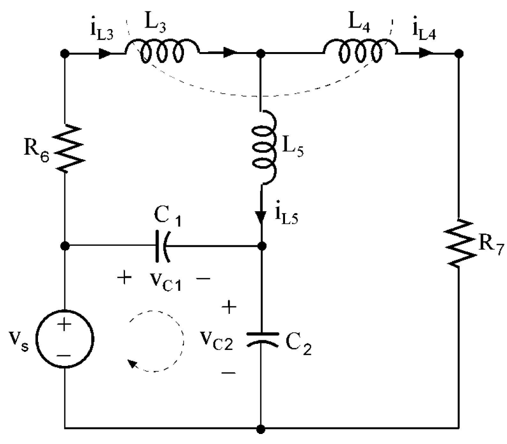

Figure 5.2 Circuit with an all-capacitor loop and an all-inductor cut-set.

(*)The opposite situation, when the circuit consists of all-inductor loops and all-capacitor cut-sets, does not influence the order of complexity, but it influences the values of the natural frequencies, namely $s = 0$. For more about all-capacitor loops/cut-sets and all-inductor cut-sets/loops see in Balabanian, N. and Bickart T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

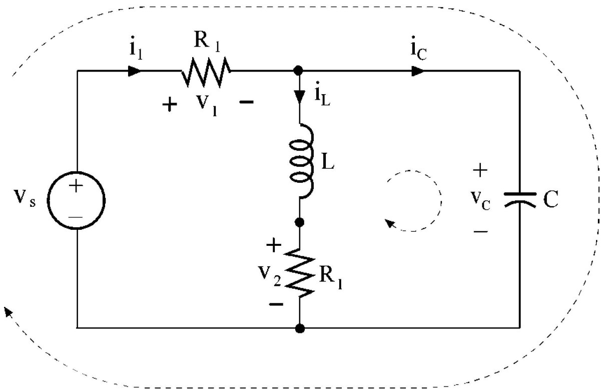

Figure 5.3 Second order circuit.

5.4 STATE EQUATIONS AND TRAJECTORY

Consider the circuit in Fig. 5.3. Let us use capacitor voltage v_C and inductor current i_L as state variables. Applying KCL to node $1n$ and KVL to the right loop and outer loop, we obtain

$$C\frac{dv_C}{dt} = -i_L + i_1, \quad L\frac{di_L}{dt} = v_C - R_2 i_L \tag{5.11}$$

$$R_1 i_1 + v_C = v_s, \tag{5.12}$$

Eliminating the non-desirable variable i_1 from equation 5.12 and substituting it into equation 5.11, after rearranging the terms, gives the state equations

$$\begin{aligned} \frac{dv_C}{dt} &= -\frac{1}{CR_1}v_C - \frac{1}{C}i_L + \frac{1}{CR_1}v_s, \\ \frac{di_L}{dt} &= \frac{1}{L}v_C - \frac{R_2}{L}i_L, \end{aligned} \tag{5.13}$$

or in *matrix form*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{w}(t), \tag{5.14}$$

where:

$\mathbf{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ is a *vector of state variables*,

$\mathbf{A} = \begin{bmatrix} -\frac{1}{CR_1} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix}$ is a *constant* 2×2 *matrix*,

$$\mathbf{b} = \begin{bmatrix} -\dfrac{1}{R_1} \\ 0 \end{bmatrix} \quad \text{is a } \textit{constant vector},$$

$\mathbf{w}(t) = v_s(t)$ is the *scalar input*, or *input vector*.

For solving equation 5.14, the initial conditions of the inductor current and of the capacitor voltage have to be known. Thus, the pair $i_L(0) = I_0$ and $v_C(0) = V_0$ is called the initial state

$$\mathbf{x}_0 = \begin{bmatrix} I_0 \\ V_0 \end{bmatrix} \tag{5.15}$$

The zero input response, i.e., circuit response when $\mathbf{w}(t) = 0$,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) \tag{5.16}$$

is completely determined by the initial state equation 5.15. Thus, if we consider $[i_L(t), v_C(t)]$ as the coordinates of a point on the $i_L - v_C$ plane, then as t increases from 0 to ∞ the point $[i_L(t), v_C(t)]$ will trace a curve, which is called the *state-space trajectory* and the plane $i_L - v_C$ is called the *state-space* of the circuit. It is obvious that the trajectory curve starts at the initial point (I_0, V_0) and ends at the origin $(0, 0)$ when $t = \infty$. Since $v_C(t)$ and $i_L(t)$ are the components of the state vector $\mathbf{x}(t)$, the trajectory defines it in the state space. The velocity of the trajectory $(di_L/dt, dv_C/dt)$ can be obtained from the state equation 5.16. In other words, the trajectory of the state vector in a two-dimensional space characterizes the behavior of a second order circuit, i.e., for every t, the corresponding point of the trajectory specifies $i_L(t)$ and $v_C(t)$.

As an example, three different kinds of trajectory, for: a) overdamped, b) underdamped and 3) loss-less, are shown in Fig. 5.4(d). Note, that in the first case, the trajectory starts at $(0.7, 0.9)$, when $t = 0$, and ends at the origin $(0, 0)$, when $t = \infty$. In the second case, the trajectory is a shrinking spiral starting at the same point and terminating at the origin. Finally, when the circuit is loss-less (which of course is an ideal circuit) the trajectory is an ellipse centered at the origin whose semi-axes depend on the circuit parameters L and C and the initial state $[i_L(0), v_C(0)]$. The ellipse shape trajectory indicates that the response is oscillatory.

For suitably chosen different initial states (usually uniformly spaced points) in the $i_L - v_C$ plane we obtain a family of trajectories, called a *phase portrait*, as shown in Fig. 5.5(a).

As we have already mentioned, the state equations in matrix representation may be easily programmed to a numerical solution. Let us illustrate the approximate method for the calculation of the trajectory. We start at the initial point, determined by the initial state $\mathbf{x}_0[v_C(0), i_L(0)]^T$, and step forward a small interval of time to find an estimate of $\mathbf{x}$ at this new time. From this point we step

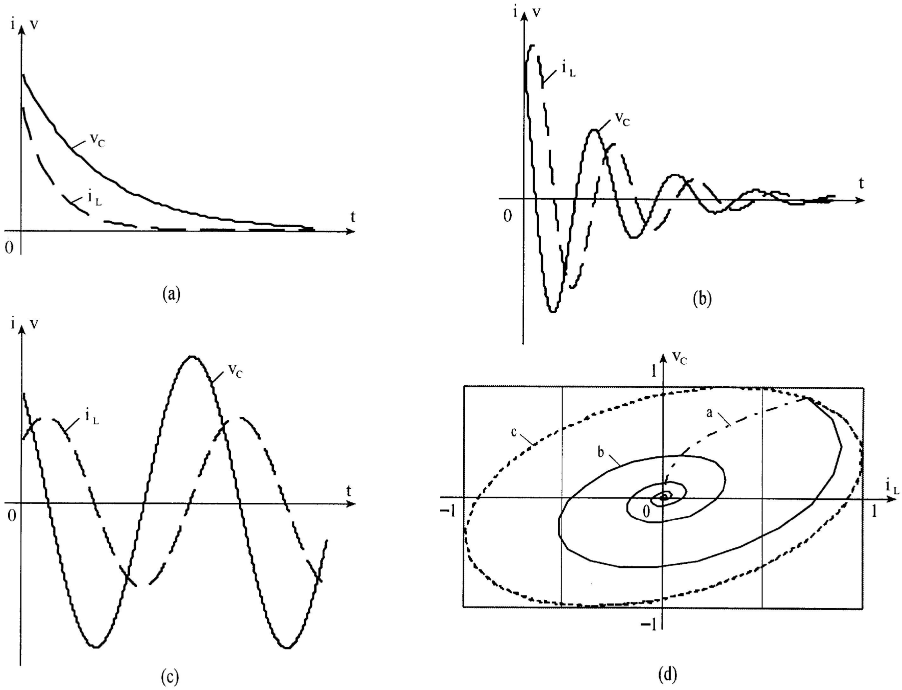

Figure 5.4 Waveforms for i_L and v_C in the second order circuits of an overdamped response (a), underdamped response (b), loss-less response (c) and state trajectories (d).

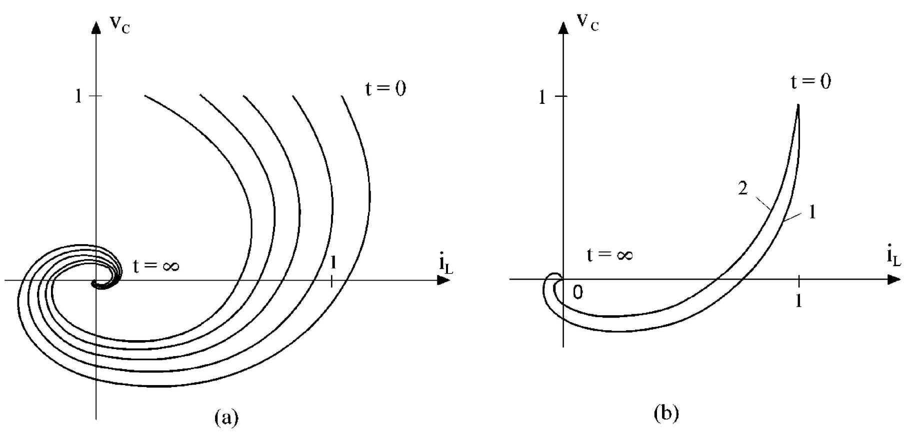

Figure 5.5 State trajectories: phase portrait (a) and for Example 5.1 (b): 1) an approximation with $\Delta t = 0.2$ s and 2) an exact trajectory.

forward again and estimate $\mathbf{x}$ after another short interval of time and so on. The estimate of $\mathbf{x}$ at the new time is found by evaluating $d\mathbf{x}/dt$ at the old time using the differential equation 5.16 and estimating the new value of $\mathbf{x}$ by the formula

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \Delta t \left(\frac{d\mathbf{x}}{dt} \right)_{old}, \tag{5.17}$$

where Δt is the "step length". This step-by-step method is known as Euler's method.

Essentially, we are using a straight-line approximation to the function in each interval. In other words, this method is based on the assumption that if a sufficiently small interval of time Δt is chosen, then during that interval the trajectory velocity $d\mathbf{x}/dt$ is approximately constant. Thus, the straight-line segment, which approximates the trajectory on each step of calculation, is

$$\Delta \mathbf{x} = \left(\frac{d\mathbf{x}}{dt} \right)_{const} \Delta t.$$

It is obvious that the approximation calculated in this manner reaches the exact trajectory when Δt approaches zero. In practice, the value of Δt that should be selected depends primarily on the accuracy required and on the length of the time interval over which the trajectory is calculated. Once the trajectory is computed, the response of the circuit is easily obtained by plotting each of the state variables v_C, i_L versus time.

Example 5.1

Let us employ Euler's (first-order) method to calculate the state trajectory and capacitor voltage versus the time of the circuit shown in Fig. 5.3.

Solution

Let the values of the circuit elements be $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$, $L = 1$ H, $C = 1$ F and the initial state be $I_0 = 1$ A and $V_0 = 1$ V.

Then, substituting the above parameters in the matrix $\mathbf{A}$, we have the state equation 5.16 as

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x},$$

and the initial state is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let us pick $\Delta t = 0.1$ s. Using equation 5.17 yields the state at 0.1 s:

$$\mathbf{x}(0.1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}.$$

Next, we can obtain the state at $t = 2\Delta t = 0.2$ s:

$$\mathbf{x}(0.2) = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.62 \\ 0.98 \end{bmatrix}.$$

From these two steps, we can write the state at $(k+1)\Delta t$ in terms of the state at $k\Delta t$

$$\mathbf{x}[(k+1)0.1] = \left(1 + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\right) \mathbf{x}(k\Delta t) = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 0.9 \end{bmatrix} \mathbf{x}(k\Delta t).$$

In accordance with this formula the computer-aided calculation results are shown in Fig. 5.5(b). If we use $\Delta t = 0.01$, the resulting trajectory will coincide with the exact trajectory.

In conclusion, the general recurrence formula for approximating the trajectory may be written as(*)

$$\mathbf{x}[(k+1)\Delta t] = (1 + \Delta t\, \mathbf{A})\mathbf{x}(k\Delta t). \tag{5.18}$$

5.5 BASIC CONSIDERATIONS IN WRITING STATE EQUATIONS

In this section, we shall introduce a systematic method for writing state equations. This method is based on the topological properties of the network and is called the "proper tree" method. However, we must first consider KCL and KVL equations based on a cut-set and loop analysis.

5.5.1 Fundamental cut-set and loop matrixes

As is known from matrix analysis, the matrix formulation of independent KCL equations is given by using the reduced incident matrix **A**. Recall that for any connected graph, having N nodes and B branches, **A** has $N-1$ rows and B columns. Thus, the set of $N-1$ linearly independent KCL equations, written on the node basis, has the matrix form

$$\mathbf{Ai} = \mathbf{0}. \tag{5.19}$$

However, equation 5.19 is not the only way of writing KCL equations. It may also be done on the cut-set basis. A cut-set is defined as a set of k branches with the property that if all k branches are removed from the graph, it is separated into two parts. As an example, consider the graph shown in Fig. 5.6.

(*)For a more accurate approximation of the state-space trajectory, the Runge-Kutta fourth-order method can be used (see, for example in Bajpai, A. C., et al. (1974) *Engineering Mathematics*, John Wiley & Sons.

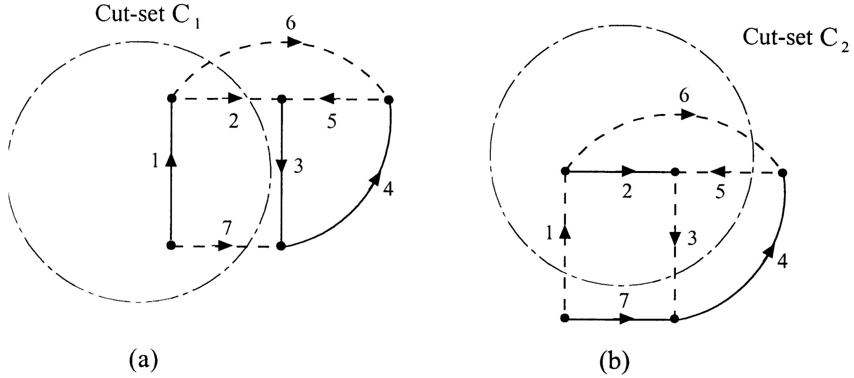

Figure 5.6 Two distinct cut-sets indicated by dashed lines.

Two distinct cut-sets are shown by dashed lines, namely $C_1 = (b_2, b_6, b_7)$ and $C_2 = (b_1, b_3, b_5, b_6)$. Recall now the generalized version of the KCL. By enclosing one of the cut parts of the circuit in the balloon-shaped surface, (see the dotted-dash line in Fig. 5.6(b)) we can write a KCL equation for this particular cut-set

$$-i_1 + i_3 - i_4 + i_5 = 0.$$

The number of such KCL equations is obviously equal to the number of distinct cut-sets. However, as we know, the number of independent KCL equations is $N-1$, where N is the number of nodes in the graph/circuit. Naturally, we are interested in writing linearly independent cut-set equations. For this purpose, we shall introduce the so-called **fundamental cut-set**. Choosing any tree in the graph, we define a fundamental cut-set as that associated with the tree branch, i.e. every tree branch together with some links constitutes a **unique cut-set** of the graph. Such a cut-set is shown, for example, in Fig. 5.7. As can be seen, removing the tree branch t_3 separates the tree into two parts T_1 and T_2. Then the links ℓ_a and ℓ_b together with twig t_3 constitute a unique cut-set. Indeed, removing any of the remaining links, even all of them (thin lines), cannot

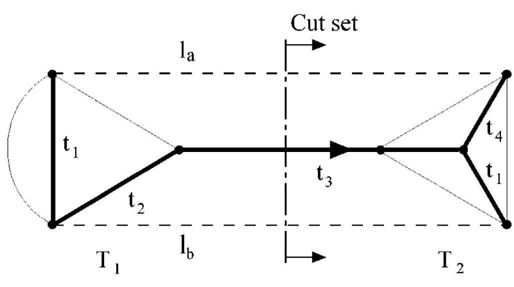

Figure 5.7 An example of a graph, tree and fundamental cut-set.

separate either T_1 or T_2 into two parts. Therefore, the above cut-set is unique. Obviously, each of the fundamental cut-sets is independent of any other, because each of them contains one and only one twig. Since the number of twigs in any tree is $N-1$, we can write $N-1$ linearly independent KCL equations following $N-1$ fundamental cut-sets. Note that the orientation of each fundamental cut-set is defined by the direction of the associated twig as shown in Fig. 5.7.

We will next consider the oriented graph of Fig. 5.8(a). A chosen tree is shown by heavy lines, and four fundamental cut-sets associated with four twigs (since a given graph has five nodes) are marked by dashed lines. For the sake of convenience, we first number the twigs from 1 to 4 and the links from 5 to 7, and adopt a reference direction for the cut-set, which agrees with the tree branch defining the cut-set. Applying KCL to the four cut-sets, we obtain

$$\begin{aligned}
&\text{cut-set 1:} \quad i_1 + i_7 = 0\\
&\text{cut-set 2:} \quad i_2 + i_6 + i_7 = 0\\
&\text{cut-set 3:} \quad i_3 - i_5 + i_6 - i_7 = 0\\
&\text{cut-set 4:} \quad i_4 - i_5 + i_6 = 0,
\end{aligned}$$

or in matrix form

$$\begin{array}{c} \textit{cut sets} \\ \\ 1\\2\\3\\4 \end{array}
\begin{array}{c}
\overbrace{\begin{array}{cccc}1&2&3&4\end{array}}^{\textit{twigs}} \quad \overbrace{\begin{array}{ccc}5&6&7\end{array}}^{\textit{links}} \\
\begin{bmatrix} 1&0&0&0&0&0&1\\ 0&1&0&0&0&1&1\\ 0&0&1&0&1&-1&-1\\ 0&0&0&1&-1&1&0 \end{bmatrix}
\end{array}
\begin{bmatrix} i_1\\ i_2\\ i_3\\ i_4\\ i_5\\ i_6\\ i_7 \end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{5.20}$$

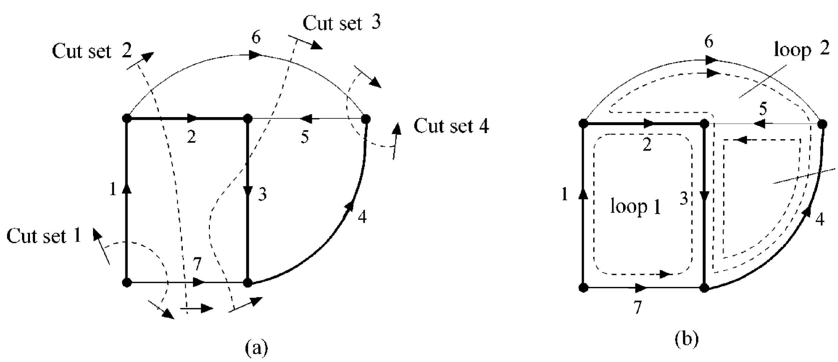

Figure 5.8 Fundamental cut-sets for the chosen tree (dashed lines) (a) and fundamental loops (dashed lines) (b).

In general, the KCL equations based on the fundamental cut-sets may be written in the short form:

$$\mathbf{Qi} = \mathbf{0}, \tag{5.21}$$

where $\mathbf{Q}$ is the *fundamental cut-set matrix* associated with the tree. The order of the $\mathbf{Q}$ matrix is $(N-1) \times B$, and its jk-th element is defined as follows:

$$q_{jk} \begin{cases} 1 & \text{if branch } k \text{ belongs to cut-set } j \text{ and has the same direction} \\ -1 & \text{if branch } k \text{ belongs to cut-set } j \text{ and has the opposite direction} \\ 0 & \text{if branch } k \text{ does not belong to cut-set } j. \end{cases}$$

Note that the fundamental cut-set matrix in equation 5.20 includes a unit sub-matrix of order $(N-1)$, which is the number of fundamental cut-sets and the number of twigs. Therefore,

$$\mathbf{Q} = [\mathbf{1}_t \quad \mathbf{Q}_\ell], \tag{5.22}$$

where $\mathbf{Q}_\ell$ is a sub-matrix of the order $(N-1) \times \ell$, i.e. it consists of $(N-1)$ rows and of ℓ (number of links) columns. The fundamental cut-set matrix $\mathbf{Q}$ will always have the form of equation 5.22 because each fundamental cut-set contains one and only one twig and its orientation agrees with the reference direction of the cut-set, by definition.

Next, we shall introduce the loop matrix. Mesh analysis, which is commonly studied in introductory courses in circuit analysis, is not the only method of writing a set of independent equations based on KVL. Another and actually more flexible method, which allows us to derive independent KVL equations, is based on the so-called **fundamental loop**. Every link of a co-tree (complement of the tree) together with some twigs, which are connected to the link, constitutes a unique loop associated with the link. Indeed, there cannot be any other path between two nodes of the tree, to which the link is connected. If there were two or more paths between two nodes of the tree, they will form a loop; this contradicts the main property of a tree. The set of fundamental loops is independent, since each of them contains one and only one link, i.e. every loop differs from another by at least one branch. Therefore, each link uniquely defines a fundamental loop. Hence, the number of fundamental loops is equal to the number of links, i.e. $B-(N-1)$. Each fundamental loop has a reference direction, which is defined by the direction of its associated link, as shown in Fig. 5.8(b).

So we use the fundamental loops to define $B-(N-1)$ linearly independent KVL equations. For the graph in Fig. 5.8(b), we may write the following three independent KVL equations:

$$\begin{array}{llllllll} \text{Loop 1:} & & & v_3 & +v_4 & +v_5 & & =0 \\ \text{Loop 2:} & & -v_2 & -v_3 & -v_4 & & +v_6 & =0 \\ \text{Loop 3:} & -v_1 & -v_2 & -v_3 & & & +v_7 & =0 \end{array}$$

or in matrix form

$$\begin{array}{c|ccccccc} \text{loops}\downarrow & \multicolumn{4}{c}{\overbrace{\hphantom{0000000000000}}^{\text{twigs}}} & \multicolumn{3}{c}{\overbrace{\hphantom{00000}}^{\text{links}}} \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.23}$$

In general, the KVL equations based on fundamental loops may be written in the short form:

$$\mathbf{Bv} = \mathbf{0}, \tag{5.24}$$

where $\mathbf{B}$ is the *fundamental loop matrix* associated with the tree. The order of the $\mathbf{B}$ matrix is $\ell \times B$, where ℓ is the number of loops, and its jk-th element is defined as follows:

$$b_{jk} \begin{cases} 1 & \text{if branch } k \text{ belongs to loop } j \text{ and has the same direction as the loop} \\ -1 & \text{if branch } k \text{ is in loop } j \text{ and has the opposite direction} \\ 0 & \text{if branch } k \text{ is not in loop } j. \end{cases}$$

Note that the fundamental loop matrix in equation 5.23 includes a unit sub-matrix of order ℓ, which is the number of fundamental loops and also the number of links. Therefore, we can express $\mathbf{B}$ in the form

$$\mathbf{B} = [\mathbf{B}_t \quad \mathbf{1}_\ell], \tag{5.25}$$

where $\mathbf{B}_t$ is a sub-matrix of $\ell \times (N-1)$, i.e. it consists of ℓ (number of links) rows and of $t = N - 1$ (number of twigs) columns. The unit matrix in $\mathbf{B}$ results from the fact that each fundamental loop contains one and only one link and by convention the reference directions of the fundamental loops are the same as that of the associated links.

Let us think that twig voltages are a set of the basic independent variables. Since each fundamental loop is formed from twigs and only one link, the link voltage can always be expressed in terms of twig voltages. Therefore, the branch voltages in any circuit can be determined by twig voltages, when the latter ones are used as independent variables. Indeed, in accordance with equations 5.24 and 5.25

$$[\mathbf{B}_t \quad \mathbf{1}_\ell] \begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_\ell \end{bmatrix} = \mathbf{0}, \tag{5.26}$$

where the branch voltage vector $\mathbf{v}$ is partitioned into two sub-vectors: $\mathbf{v}_t$ and

$\mathbf{v}_\ell$, which are, respectively, the twig-voltage sub-vector and link-voltage sub-vector. Performing the multiplication yields

$$\mathbf{B}_t\mathbf{v}_t + \mathbf{v}_\ell = \mathbf{0},$$

or

$$\mathbf{v}_\ell = -\mathbf{B}_t\mathbf{v}_t. \tag{5.27}$$

This means that link voltages are determined by twig voltages. Obviously, we can write the twig branch-voltage sub-vector as

$$\mathbf{v}_t = \mathbf{1}_t\mathbf{v}_t. \tag{5.28}$$

Combining equations 5.27 and 5.28, we have

$$\begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_\ell \end{bmatrix} = \begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t, \tag{5.29}$$

or simply

$$\mathbf{v} = \begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t, \tag{5.30}$$

which states that all the branch voltages in any circuit can be expressed in terms of twig voltages.

Now, let us again examine the fundamental cut-sets. Since each fundamental cut-set is formed from links and only one twig, we can express the twig-currents in terms of link-currents. Therefore, using the link-currents as basic independent variables, we can always determine the all branch currents by the independent variables. After partitioning the branch currents into twig-currents and link-currents, with equations 5.21 and 5.22, we have

$$[\mathbf{1}_t \quad \mathbf{Q}_\ell] \begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_\ell \end{bmatrix} = \mathbf{0}, \tag{5.31}$$

where $\mathbf{i}_t$ and $\mathbf{i}_\ell$ are, respectively, the twig-current and link-current sub-vectors. Then two matrixes in equation 5.31 can be multiplied to yield

$$\mathbf{i}_t + \mathbf{Q}_\ell\mathbf{i}_\ell = \mathbf{0},$$

or

$$\mathbf{i}_t = -\mathbf{Q}_\ell\mathbf{i}_\ell. \tag{5.32}$$

Combining equation 5.32 and the identity $\mathbf{i}_\ell = \mathbf{1}_\ell\mathbf{i}_\ell$, yields

$$\begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_\ell \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}_\ell \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell, \tag{5.33}$$

or

$$\mathbf{i} = \begin{bmatrix} -\mathbf{Q}_\ell \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell, \tag{5.34}$$

which again states that all branch currents in any circuit can be expressed in terms of link currents.

A useful relation between two matrixes $\mathbf{Q}$ and $\mathbf{B}$ can now be determined. Recall *Tellegen's theorem* in the form

$$\mathbf{v}^T \mathbf{i} = \mathbf{0}. \tag{5.35}$$

By taking the transpose of $\boldsymbol{v}$ (equation 5.30), we obtain

$$\mathbf{v}^T = \left(\begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t \right)^T = \mathbf{v}_t^T \begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix}^T = \mathbf{v}^T [\mathbf{1}_t - \mathbf{B}_t^T]. \tag{5.36}$$

After substituting equations 5.36 and 5.34 into equation 5.35 we have

$$\mathbf{v}_t^T [\mathbf{1}_t - \mathbf{B}_t^T] \begin{bmatrix} -\mathbf{Q}_t \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell = \mathbf{0}, \quad \text{for all } \mathbf{v}_t \text{ and all } \mathbf{i}_\ell. \tag{5.37}$$

Since $\mathbf{v}_t^T \neq \mathbf{0}$ and $\mathbf{i}_\ell \neq \mathbf{0}$ then

$$[\mathbf{1}_t - \mathbf{B}_t^T] \begin{bmatrix} -\mathbf{Q}_t \\ \mathbf{1}_\ell \end{bmatrix} = \mathbf{0}. \tag{5.38}$$

Performing the multiplication, we obtain the identities

$$\mathbf{Q}_\ell = -\mathbf{B}_t^T \tag{5.39a}$$

and

$$\mathbf{B}_t = -\mathbf{Q}_\ell^T. \tag{5.39b}$$

This relationship between two sub-matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ results from the fact that both fundamental cut-set matrix $\mathbf{Q}_\ell$ and fundamental loop matrix $\mathbf{B}_t$ give the topological relation between graph branches and fundamental cut-sets and fundamental loops respectively. Also, note that both matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ come from the same tree.

Replacing $-\mathbf{B}_t$ by $\mathbf{Q}_\ell^T$ in equation 5.30, we obtain

$$\mathbf{v} = \begin{bmatrix} \mathbf{1}_t \\ \mathbf{Q}_\ell^T \end{bmatrix} \mathbf{v}_t = \mathbf{Q}^T \mathbf{v}_t, \tag{5.40}$$

which can be interpreted as a matrix transformation of twig-voltages into branch voltages. Similarly, replacing $-\mathbf{Q}_\ell$ by $\mathbf{B}_t^T$ in equation 5.34, we obtain

$$\mathbf{i} = \begin{bmatrix} \mathbf{B}_t^T \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell = \mathbf{B}^T \mathbf{i}_\ell, \tag{5.41}$$

which is a matrix transformation of link-currents into branch currents.

Finally, substituting equations 5.40 and 5.41 into Tellengen's theorem (equation 5.35), we have

$$\mathbf{v}_t^T \mathbf{Q} \mathbf{B}^T \mathbf{i}_\ell = \mathbf{0}, \quad \text{for all } \mathbf{v}_t \text{ and } \mathbf{i}_\ell, \tag{5.42}$$

which can be reduced to the following relation between the matrixes

$$\mathbf{Q}\mathbf{B}^T = \mathbf{0}. \tag{5.43}$$

In conclusion, the following comments on loop and cut-set matrixes have to be made. The methods of circuit analysis based on loop and cut-set matrixes are more flexible, allowing more possible applications than the node and mesh analyses. So, as we remember, the mesh analysis based on mesh matrix **M** is restricted to the planar graph only, whereas the fundamental loop matrix **B**, based on tree, is applicable to any graph including non-planar, by means of allowing us to write a maximal number of linearly independent KVL equations.

The concept of duality is usually applied (in introductory courses) to planar graphs and planar circuits by means of node and mesh terms. By now, we may extend this concept to fundamental matrixes **B** and **Q**, pertaining to non-planar graphs and circuits. So, the listing of dual terms can be extended as follows:

Twig	– Link,
Fundamental cut-set	– Fundamental loop,
Twig voltage, v_t	– Link current, i_ℓ,
Fundamental cut-set matrix, **Q**	– Fundamental loop matrix, **B.**

Thus, two graphs, G_1 and G_2 having the same number of branches, are dual if the number of fundamental cut-sets of one of them is equal to the number of fundamental loops of the second and their **Q** and **B** matrixes are identical, namely

$$\mathbf{Q}_1 = \mathbf{B}_2.$$

5.5.2 "Proper tree" method for writing state equations

Our aim now is to write the state and output equations in the form of equation 5.9

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{w}(t) \tag{5.44a}$$

$$\mathbf{y}(t) = \mathbf{c}\mathbf{x}(t) + \mathbf{d}\mathbf{w}(t), \tag{5.44b}$$

where **x** is the state vector containing all the capacitor voltages and all the inductor currents, **w** is the input vector containing all the independent voltage and current sources, driving the circuit and **y** is the desired output vector. **A**, **b**, **c** and **d** are constant matrixes whose elements depend on circuit parameters. Equation 5.44a is a first order matrix differential equation with constant matrix coefficients. $\dot{\mathbf{x}}$ is the first derivative of the state vector **x**, i.e. it consists of the derivatives of the state variables dv_C/dt and di_L/dt. We note that these quantities are given by currents in the capacitors $C(dv_C/dt)$ and voltages across inductors $L(di_L/dt)$. To evaluate capacitor currents in terms of other currents, we must write cut-set equations and to evaluate inductor voltages in terms of other voltages we must write loop equations. Therefore, it turns out that we could do this if, using the concept of cut-set and loop analysis, we chose a tree which

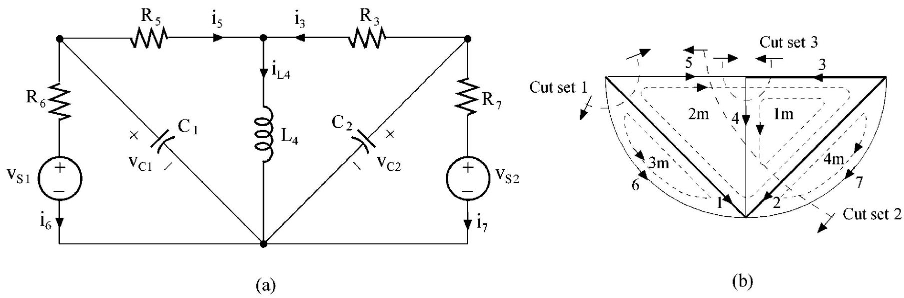

Figure 5.9 A circuit of the example for writing state equations (a), the oriented graph and proper tree (b).

includes all the capacitors but no inductors. Such a tree is called a *proper tree*(*) We can complete the proper tree if the number of twigs is larger than the number of capacitors by adding resistors and voltage sources. Thus, the inductors, the remaining resistors and possibly the current sources will constitute the co-tree links.

Following this method, we may write a fundamental cut-set equation for each capacitor-twig, in which the capacitor current $C(dv_C/dt)$ is expressed in terms of other currents. We may write a fundamental loop equation as well for each link inductor in which the inductor voltage $L(di_L/dt)$ is expressed in terms of other voltages. We shall also take into consideration that the basic variables in cut-set/loop analysis are twig voltages and link currents. Hence, we shall use the appropriate v–i relationships for resistive and active elements. Thus for twig resistors we use the form $v_t = Ri$ and for the link resistors $i_\ell = Gv$. For the same reason we put the voltage sources into the twigs and the current sources into the links. (To fulfill these requirements, we can use a source transformation and shifting techniques.) At this point, let us illustrate the above description by the following example. For the sake of generality, we will divide the solution procedure into five steps. Consider the circuit shown in Fig. 5.9(a).

Step 1 Choosing the state variables

The circuit contains two capacitors and one inductor. Therefore, the state variables are v_{C1}, v_{C2} and i_{L4}, and the state vector is

$$\mathbf{x} = \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix}. \tag{5.45}$$

Step 2 Choosing the proper tree

(*)If a circuit contains an all-capacitor loop or an all-inductor cut-set, a proper tree does not exist. For such cases see in Balabanian, N. and Bickart, T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

The proper tree picked for the circuit, shown in Fig. 5.9(b), includes two capacitors and resistor R_3.

Step 3 Writing the fundamental cut-set equations

These equations are written in such a way that the capacitor currents are defined by other link currents and/or current sources (if such are present), and the remaining currents are written in terms of inductor currents and/or current sources.

$$\begin{aligned}
&\text{cut-set 1:} \quad C_1 \frac{dv_{C1}}{dt} = -i_5 - i_6 \\
&\text{cut-set 2:} \quad C_2 \frac{dv_{C2}}{dt} = -i_{L4} + i_5 - i_7
\end{aligned} \tag{5.46}$$

$$\text{cut-set 3:} \quad G_3 v_3 + i_5 = i_{L4}. \tag{5.47}$$

Step 4 Writing the fundamental loop equations

The loop equations are written in such a way that the inductor voltages are defined by other twig voltages and/or voltage sources (if such are present), and the remaining voltages are written in terms of capacitor voltages and/or voltage sources

$$\text{Loop 1:} \quad L_4 \frac{di_{L4}}{dt} = v_{C2} - v_3 \tag{5.48}$$

$$\text{Loop 2:} \quad -v_3 + R_5 i_5 = v_{C1} - v_{C2} \tag{5.49}$$

$$\left.\begin{aligned}
&\text{Loop 3:} \quad R_6 i_6 = v_{C1} - v_{s1} \\
&\text{Loop 4:} \quad R_7 i_7 = v_{C2} - v_{s2}
\end{aligned}\right\}. \tag{5.50}$$

The last two steps lead to state equations

$$\begin{aligned}
C_1 \frac{dv_{C1}}{dt} &= -i_5 - i_6 \\
C_2 \frac{dv_{C2}}{dt} &= -i_{L4} + i_5 - i_7 \\
L_4 \frac{di_{L4}}{dt} &= v_{C2} - v_3.
\end{aligned} \tag{5.51}$$

Step 5 Expressing the right-hand side of the state equations in terms of state variables and/or inputs. In this example, currents i_5, i_6, i_7 and voltage v_3 have to be expressed in terms of the capacitor voltages v_{C1}, v_{C2} and the inductor current i_{L4}. By solving equations 5.50, we have

$$i_6 = \frac{1}{R_6} v_{C1} - \frac{1}{R_6} v_{s1}, \quad i_7 = \frac{1}{R_7} v_{C2} - \frac{1}{R_7} v_{s2}, \tag{5.52}$$

equations 5.47 and 5.49 form a set of two algebraic equations of two unknowns:

$$\begin{bmatrix} -1 & R_5 \\ G_3 & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_{C1} - v_{C2} \\ i_{L4} \end{bmatrix} \tag{5.53}$$

Solving equation 5.53 yields

$$\begin{aligned} v_3 &= -\frac{1}{1+R_5G_3} v_{C1} + \frac{1}{1+R_5G_3} v_{C2} + \frac{R_5}{1+R_5G_3} i_{L4} \\ i_5 &= \frac{G_3}{1+R_5G_3} v_{C1} - \frac{G_3}{1+R_5G_3} v_{C2} + \frac{1}{1+R_5G_3} i_{L4}. \end{aligned} \tag{5.54}$$

Finally, equations 5.52 and 5.54 can be substituted into equation 5.51 to yield, after rearrangement and dividing through the equations by appropriate C_1, C_2, L_4,

$$\frac{d}{dt}\begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} = \begin{bmatrix} -\frac{1+aR_6G_3}{R_6C_1} & \frac{aG_3}{C_1} & -\frac{a}{C_1} \\ \frac{aG_3}{C_2} & -\frac{1+aR_7G_3}{R_7C_2} & -\frac{1-a}{C_2} \\ \frac{a}{L_4} & \frac{1-a}{L_4} & -\frac{aR_5}{L_4} \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_6C_1} & 0 \\ 0 & \frac{1}{R_7C_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix}, \tag{5.55}$$

where $a = 1/(1+R_5G_3)$.

Note that state equations here are written in the matrix form of equation 5.44a where the input vector (in this example) is $\mathbf{w} = [v_{s1} \;\; v_{s2}]^T$ and the meanings of matrixes $\mathbf{A}$ and $\mathbf{b}$ are obvious.

Suppose now that the remaining branch variables, i.e. v_3, i_5, i_6 and i_7 are a desired output. Then, using equations 5.54 and 5.52, we can express the output in terms of the state variables and the input as

$$\begin{bmatrix} v_3 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} -a & a & aR_5 \\ aG_3 & -aG_3 & a \\ 1/R_6 & 0 & 0 \\ 0 & 1/R_7 & 0 \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/R_6 & 0 \\ 0 & -1/R_7 \end{bmatrix} \begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix}. \tag{5.56}$$

This is an output equation in the form of equation 5.44b, where the output vector is $\mathbf{y} = [v_3 \;\; i_5 \;\; i_6 \;\; i_7]^T$ and the meanings of the constant matrixes are obvious.

Remark. The capacitor charges and the inductor fluxes can also be used as state variables. Then in the above example the state vector will be

$$\mathbf{x} = [q_1 \quad q_2 \quad \lambda_4]^T,$$

where $q_1 = C_1 v_{C1}$, $q_2 = C_2 v_{C2}$ and $\lambda_4 = L_4 i_{L4}$.

Substituting $v_{C1} = q_1/C_1$, $v_{C2} = q_2/C_2$ and $i_4 = \lambda_4/L_4$ in equation 5.55, and after simplification, we obtain

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} -\dfrac{1+aR_6G_3}{R_6C_1} & \dfrac{aG_3}{C_2} & -\dfrac{a}{L_4} \\ \dfrac{aG_3}{C_1} & -\dfrac{1+aR_7G_3}{R_7C_2} & -\dfrac{1+a}{L_4} \\ \dfrac{a}{C_1} & \dfrac{1-a}{C_2} & -\dfrac{aR_5}{L_4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_{C2} \\ \lambda_4 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_6} & 0 \\ 0 & \dfrac{1}{R_7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix} \tag{5.57}$$

which is the state equation using the charges and fluxes as state variables.

It is worthwhile mentioning that some other variables in the circuit may be used as state variables. For example, a current through a resistor in parallel with a capacitor or voltage across a resistor in series with an inductor can be treated as state variables. Also any linear combination of capacitor voltages or inductor currents may be used as state variables. This can be helpful in writing state equations when the circuit consists of all-capacitor loops or all-inductor cut-sets. The next step would be to solve the state equations. However, before doing so, we shall consider the general approach for deriving state equations in matrix form.

5.6 A SYSTEMATIC METHOD FOR WRITING A STATE EQUATION BASED ON CIRCUIT MATRIX REPRESENTATION

Consider a network whose elements are inductors, capacitors, resistors and independent sources. As stated, we assume that capacitors do not form a loop and inductors do not form a cut-set. We also assume that the network graph is connected and as a first step we will pick a *proper tree*. We can obviously include all capacitors into the tree branches, since they do not form any loop. Usually, it might be necessary to add some resistors and/or voltage sources in order to complete the tree. Then all the inductors will be assigned to the links. In the next step we shall partition the circuit branches into four sub-sets: the capacitive twigs, the resistive twigs, the inductive links and the resistive links. For the sake of specifics, we shall use an example to illustrate this procedure.

Consider again the circuit shown in Fig. 5.9(a). The circuit graph and the proper tree are shown in Fig. 5.9(b). The KCL equations for the fundamental cut-sets, in accordance with equation 5.31, are

$$[\mathbf{1}_t \quad \mathbf{Q}_\ell]\begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \\ -- \\ \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} = \mathbf{0}, \tag{5.58}$$

where subvectors of twig and link currents are

$$\mathbf{i}_t = \begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \end{bmatrix}, \quad \mathbf{i}_\ell = \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix}$$

and $\mathbf{i}_C$, $\mathbf{i}_G$, $\mathbf{i}_L$ and $\mathbf{i}_R$ are in turn subvectors representing currents in capacitive and resistive (conductive) twigs and inductive and resistive links, respectively. In our example, these four subvectors are

$$\mathbf{i}_C = \begin{bmatrix} i_{C1} \\ i_{C2} \end{bmatrix}, \quad \mathbf{i}_G = [i_{G3}], \quad \mathbf{i}_L = [i_{L4}], \quad \mathbf{i}_R = \begin{bmatrix} i_{R5} \\ i_{R6} \\ i_{R7} \end{bmatrix} \tag{5.59}$$

and the equation 5.58 becomes

$$\left[\begin{array}{ccc:c:ccc} & & & \overbrace{}^{\mathbf{Q}_{CL}} & & \overbrace{}^{\mathbf{Q}_{CR}} & \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ \hdashline 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ & & & \underbrace{}_{\mathbf{Q}_{GL}} & & \underbrace{}_{\mathbf{Q}_{GR}} & \end{array}\right] \begin{bmatrix} i_{C1} \\ i_{C2} \\ i_{C3} \\ \hdashline i_{L4} \\ i_{R5} \\ i_{R6} \\ i_{R7} \end{bmatrix} = \mathbf{0} \tag{5.60}$$

$$\underbrace{}_{\mathbf{Q}_\ell}$$

The KVL equations may be written in the form (see equation 5.26)

$$[\mathbf{B}_t \quad \mathbf{1}_\ell]\begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \\ -- \\ \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix} = \mathbf{0}, \tag{5.61}$$

where

$$\mathbf{v}_t = \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix}, \quad \mathbf{v}_\ell = \begin{bmatrix} \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix}$$

are subvectors of twig and link voltages and $\mathbf{v}_C$, $\mathbf{v}_G$, $\mathbf{v}_L$, $\mathbf{v}_R$ are in turn subvectors representing voltages across the capacitive and resistive (conductive) twigs and inductive and resistive links, respectively. For the circuit in Fig. 5.9(a) the voltage subvectors are

$$\mathbf{v}_C = \begin{bmatrix} v_{C1} \\ v_{C2} \end{bmatrix}, \quad \mathbf{v}_G = [v_{G3}], \quad \mathbf{v}_L = [v_{L4}], \quad \mathbf{v}_R = \begin{bmatrix} v_{R5} \\ v_{R6} - v_{sR6} \\ v_{R7} - v_{sR7} \end{bmatrix} = \begin{bmatrix} v_\ell 5 \\ v_6 \\ v_7 \end{bmatrix} \tag{5.62}$$

where v_{sR6} represents v_{s1} and v_{sR7} represents v_{s2}. The KVL equation 5.61 becomes

$$\left[\begin{array}{cc:c:cccc} \multicolumn{2}{c}{\overbrace{\quad\quad}^{\mathbf{B}_{LC}}} & \overbrace{\quad}^{\mathbf{B}_{LG}} & & & & \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ \hdashline -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ \multicolumn{2}{c}{\underbrace{\quad\quad}_{\mathbf{B}_{RC}}} & \underbrace{\quad}_{\mathbf{B}_{RG}} & & & & \end{array}\right] \begin{bmatrix} v_{C1} \\ v_{C2} \\ v_{G3} \\ \hdashline v_{L4} \\ v_{R5} \\ v_6 \\ v_7 \end{bmatrix} = \mathbf{0}. \tag{5.63}$$

where the first three columns form $\mathbf{B}_t$.

Note that $\mathbf{B}_t = -\mathbf{Q}_\ell^T$.

Next we shall use the v-i, or i-v characteristics to introduce branch equations. We will employ the concept of a generalized branch, i.e. combining passive and active elements together. However, we must now take into consideration four different branches: two for twigs and two for links, as shown in Fig. 5.10. As was mentioned earlier, we shall assume that the voltage sources are located in the link branches and the current sources are located in the twig branches. Therefore, in matrix form we have:

$$\begin{aligned} &\textit{capacitor twigs} \quad \mathbf{i}_C = \mathbf{C}\frac{d}{dt}\mathbf{v}_C + \mathbf{i}_{sC} \\ &\textit{inductor links} \quad \mathbf{v}_L = \mathbf{L}\frac{d}{dt}\mathbf{i}_L + \mathbf{v}_{sL} \end{aligned} \tag{5.64}$$

$$\begin{aligned} &\textit{resistor twigs} \quad \mathbf{i}_G = \mathbf{G}\mathbf{v}_G + \mathbf{i}_{sG} \\ &\textit{resistor links} \quad \mathbf{v}_R = \mathbf{R}\mathbf{i}_R + \mathbf{v}_{sR} \end{aligned} \tag{5.65}$$

where the matrixes **C**, **L**, **G** and **R** are the branch parameter matrixes; namely,

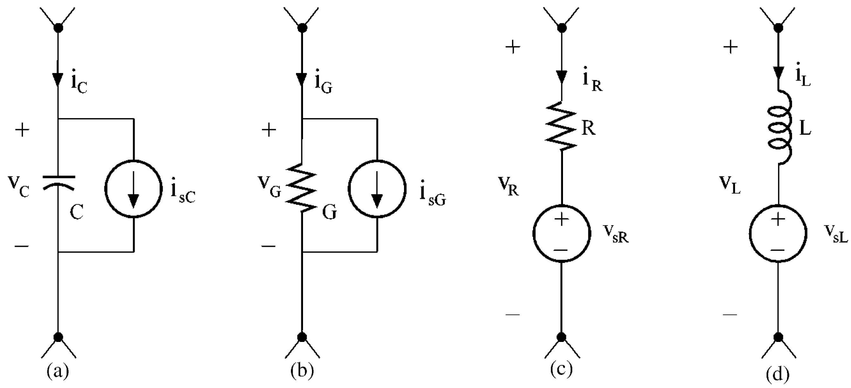

Figure 5.10 Generalized branches with independent sources: twig capacitor (a), twig resistor (b), link resistor (c) and link inductor (d).

the twig capacitance matrix, the link inductance matrix, the twig conductance matrix and the link resistance matrix, respectively. Note that **C**, **L**, **G** and **R** are square diagonal matrixes, but if the circuit consists of coupled elements (mutual inductances and/or dependent sources), **L**, **G** and **R** might not be diagonal any more. For the example in Fig. 5.9

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad \mathbf{L} = [L_4] \tag{5.66}$$

$$\mathbf{G} = [G_3], \quad \mathbf{R} = \begin{bmatrix} R_5 & & 0 \\ & R_6 & \\ 0 & & R_7 \end{bmatrix}. \tag{5.67}$$

The vectors $\mathbf{v}_{sR}$, $\mathbf{v}_{sL}$ and $\mathbf{i}_{sG}$, $\mathbf{i}_{sC}$ represent the independent voltage and current sources, which in the present example are

$$\mathbf{v}_{sR} = \begin{bmatrix} 0 \\ v_{s1} \\ v_{s2} \end{bmatrix}, \quad \mathbf{v}_{sL} = \mathbf{0}, \quad \mathbf{i}_{sG} = \mathbf{0}, \quad \mathbf{i}_{sC} = \mathbf{0}. \tag{5.68}$$

Equation 5.64 can be rewritten to yield

$$\mathbf{C}\frac{d}{dt}\mathbf{v}_C = \mathbf{i}_C - \mathbf{i}_{sC}, \quad \mathbf{L}\frac{d}{dt}\mathbf{i}_L = \mathbf{v}_L - \mathbf{v}_{sL}. \tag{5.69}$$

To bring these equations to the form of state equations, we must eliminate the variables. For this purpose, we shall solve the KCL equation 5.58 and KVL equation 5.61 equations together with the branch equations 5.64 and 5.65.

Equations 5.58 and 5.61 can be rewritten as

$$\begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \end{bmatrix} = -\mathbf{Q}_\ell \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} = -\begin{bmatrix} \mathbf{Q}_{CL} & \mathbf{Q}_{CR} \\ \mathbf{Q}_{GL} & \mathbf{Q}_{GR} \end{bmatrix} \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} \tag{5.70a}$$

and

$$\begin{bmatrix} \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix} = -\mathbf{B}_t \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix} = -\begin{bmatrix} \mathbf{B}_{LC} & \mathbf{B}_{LG} \\ \mathbf{B}_{RC} & \mathbf{B}_{RG} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix} \tag{5.70b}$$

where in the following solution matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ are partitioned into submatrixes. The order of each of the submatrixes in equations 5.70 is determined by the number of twigs (which is the number of rows) and by the number of corresponding links (which is the number of columns) in equation 5.70a and vice versa in equation 5.70b. For example, the number of rows in $\mathbf{Q}_{CL}$ (equation 5.70a) is equal to the number of capacitor currents in $\mathbf{i}_C$ (capacitor twigs) and the number of its columns is equal to the number of inductor currents in $\mathbf{i}_L$ (inductor links). It can also be shown that there are simple relations between $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ submatrixes, namely

$$\mathbf{B}_{LC} = -\mathbf{Q}_{CL}^T, \quad \mathrm{B}_{LG} = -\mathbf{Q}_{GL}^T, \quad \mathbf{B}_{RC} = -\mathbf{Q}_{CR}^T, \quad \mathbf{B}_{RG} = -\mathbf{Q}_{GR}^T. \tag{5.71}$$

The undesirable variables $\mathbf{i}_C$ and $\mathbf{v}_L$ in equation 5.69 can now be expressed from equation 5.70 to yield

$$\mathbf{i}_C = -\mathbf{Q}_{CL}\mathbf{i}_L - \mathbf{Q}_{CR}\mathbf{i}_R \tag{5.72a}$$

$$\mathbf{v}_L = -\mathbf{B}_{LC}\mathbf{v}_C - \mathbf{B}_{LG}\mathbf{v}_G, \tag{5.72b}$$

and after substituting these two expressions into equation 5.69, we obtain

$$\begin{aligned} \mathbf{C}\frac{d}{dt}\mathbf{v}_C &= -\mathbf{Q}_{CL}\mathbf{i}_L - \mathbf{Q}_{CR}\mathbf{i}_R - \mathbf{i}_{sC} \\ \mathbf{L}\frac{d}{dt}\mathbf{i}_L &= -\mathbf{B}_{LC}\mathbf{v}_C - \mathbf{B}_{LG}\mathbf{v}_G - \mathbf{v}_{sL}. \end{aligned} \tag{5.73}$$

However, we still need to eliminate $\mathbf{i}_R$ and $\mathbf{v}_G$. Substituting $\mathbf{i}_G$ and $\mathbf{v}_R$ from equation 5.70 into equation 5.65, and after rearrangement, results in two simultaneous matrix equations in two unknowns $\mathbf{i}_R$ and $\mathbf{v}_G$,

$$\mathbf{R}\mathbf{i}_R + \mathbf{B}_{RG}\mathbf{v}_G = \mathbf{M} \tag{5.73a}$$

$$\mathbf{Q}_{GR}\mathbf{i}_R + \mathbf{G}\mathbf{v}_G = \mathbf{N}, \tag{5.73b}$$

where

$$\mathbf{M} = -\mathbf{B}_{RC}\mathbf{v}_C - \mathbf{v}_{sR} \quad \text{and} \quad \mathbf{N} = -\mathbf{Q}_{GL}\mathbf{i}_L - \mathbf{i}_{sG} \tag{5.74}$$

Solving these two equations by the substitution method yields

$$\mathbf{i}_R = \mathbf{R}_{eq}^{-1}(-\mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{N} + \mathbf{M}) \tag{5.75a}$$

$$\mathbf{v}_G = \mathbf{G}_{eq}^{-1}(-\mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{M} + \mathbf{N}), \tag{5.75b}$$

where

$$\mathbf{R}_{eq} = \mathbf{R} - \mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{Q}_{GR} \tag{5.76a}$$

$$\mathbf{G}_{eq} = \mathbf{G} - \mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{B}_{RG}. \tag{5.76b}$$

Finally, we substitute equation 5.75 with equation 5.74 in equation 5.73 to obtain, after rearrangement, the state representation is follows

$$\frac{d}{dt}\begin{bmatrix}\mathbf{v}_C\\ \mathbf{i}_L\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}}_{\mathbf{A}} \underbrace{\begin{bmatrix}\mathbf{A}_{11}^1 & \mathbf{A}_{12}^1\\ \mathbf{A}_{21}^1 & \mathbf{A}_{22}^1\end{bmatrix}}_{\mathbf{A}^1}\begin{bmatrix}\mathbf{v}_C\\ \mathbf{i}_L\end{bmatrix}$$

$$+\underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}}_{\mathbf{b}} \underbrace{\begin{bmatrix}\mathbf{b}_{11}^1 & \mathbf{b}_{12}^1 & \mathbf{b}_{13}^1 & \mathbf{b}_{14}^1\\ \mathbf{b}_{21}^1 & \mathbf{b}_{22}^1 & \mathbf{b}_{23}^1 & \mathbf{b}_{24}^1\end{bmatrix}}_{\mathbf{b}^1}\begin{bmatrix}\mathbf{i}_{sC}\\ \mathbf{i}_{sG}\\ \mathbf{v}_{sL}\\ \mathbf{v}_{sR}\end{bmatrix} \tag{5.77}$$

where the matrix terms are

$$\begin{aligned}&\mathbf{A}_{11}^1 = \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RC} \quad \mathbf{A}_{12}^1 = -\mathbf{Q}_{CL} - \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{Q}_{GL}\\ &\mathbf{A}_{22}^1 = \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GL} \quad \mathbf{A}_{21}^1 = -\mathbf{B}_{LC} - \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{B}_{RC}\end{aligned} \tag{5.78}$$

$$\begin{aligned}&\mathbf{b}_{11}^1 = -\mathbf{1} \quad \mathbf{b}_{12}^1 = -\mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RG}\mathbf{G}^{-1} \quad \mathbf{b}_{13}^1 = \mathbf{0} \quad \mathbf{b}_{14}^1 = \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\\ &\mathbf{b}_{21}^1 = \mathbf{0} \quad \mathbf{b}_{22}^1 = \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1} \quad \mathbf{b}_{23}^1 = -\mathbf{1} \quad \mathbf{b}_{24}^1 = -\mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GR}\mathbf{R}^{-1}.\end{aligned} \tag{5.79}$$

Let us now use the above expressions to calculate the **A** and **b** matrixes in our example.

First we determine the submatrixes of the $\mathbf{Q}_\ell$ matrix

$$\mathbf{Q}_\ell\begin{bmatrix}\mathbf{Q}_{CL} & \mathbf{Q}_{CR}\\ \mathbf{Q}_{GL} & \mathbf{Q}_{GR}\end{bmatrix} = \left[\begin{array}{c|ccc}0 & 1 & 1 & 0\\ 1 & -1 & 0 & 1\\ \hline -1 & 1 & 0 & 0\end{array}\right].$$

Then with equation 5.76 and equation 5.71 we have

$$\mathbf{R}_{eq}\begin{bmatrix}R_5 & & \mathbf{0}\\ & R_6 & \\ \mathbf{0} & & R_7\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\left[\frac{1}{G_3}\right][1 \quad 0 \quad 0] = \begin{bmatrix}\dfrac{1+R_5G_3}{G_3} & 0 & 0\\ 0 & R_6 & 0\\ 0 & 0 & R_7\end{bmatrix}$$

$$\mathbf{R}_{eq}^{-1} = \begin{bmatrix}aG_3 & 0 & 0\\ 0 & 1/R_6 & 0\\ 0 & 0 & 1/R_7\end{bmatrix}, \quad \text{where again } a = \frac{1}{1+R_5G_3}$$

$$\mathbf{G}_{eq}=[G_3]+[1 \quad 0 \quad 0]\begin{bmatrix}1/R_5 & 0 & 0\\ 0 & 1/R_6 & 0\\ 0 & 0 & 1/R_7\end{bmatrix}\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}=\left[\frac{1+R_5G_3}{R_5}\right]=\left[\frac{1}{aR_5}\right]$$

$$\mathbf{G}_{eq}^{-1}=[aR_5]$$

$$\mathbf{A}_{11}^{1}=\begin{bmatrix}1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}aG_3 & & 0\\ & 1/R_6 & \\ 0 & & 1/R_7\end{bmatrix}\begin{bmatrix}-1 & +1\\ -1 & 0\\ 0 & -1\end{bmatrix}$$

$$=-\begin{bmatrix}(aG_3+1/R_6) & -aG_3\\ -aG_3 & (aG_3+1/R_7)\end{bmatrix}$$

$$\mathbf{A}_{22}^{1}=[1][aR_5][-1]=-[aR_5]$$

$$\mathbf{A}_{12}^{1}=-\begin{bmatrix}0\\ 1\end{bmatrix}-\begin{bmatrix}1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}aG_3 & 0 & 0\\ 0 & 1/R_6 & 0\\ 0 & 0 & 1/R_7\end{bmatrix}\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}[1/G_3][-1]$$

$$=-\begin{bmatrix}1\\ 1-a\end{bmatrix}$$

$$\mathbf{A}_{21}^{1}=-[0 \quad -1]-[1][aR_5][1 \quad 0 \quad 0]\begin{bmatrix}1/R_5 & 0 & 0\\ 0 & 1/R_6 & 0\\ 0 & 0 & 1/R_7\end{bmatrix}\begin{bmatrix}-1 & +1\\ -1 & 0\\ 0 & -1\end{bmatrix}$$

$$=[a \quad (1-a)]$$

$$\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}=\begin{bmatrix}C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & L_2\end{bmatrix}^{-1}=\begin{bmatrix}1/C_1 & 0 & 0\\ 0 & 1/C_2 & 0\\ 0 & 0 & 1/L_4\end{bmatrix}$$

Therefore the **A** matrix is

$$\mathbf{A}=\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_{11}^{1} & \mathbf{A}_{12}^{1}\\ \mathbf{A}_{21}^{1} & \mathbf{A}_{22}^{1}\end{bmatrix}=\left[\begin{array}{cc|c}-\dfrac{1+R_6aG_3}{R_6C_1} & \dfrac{aG_3}{C_1} & -\dfrac{a}{C_1}\\ \dfrac{aG_3}{C_2} & -\dfrac{1+R_7aG_3}{R_7C_2} & -\dfrac{1-a}{C_2}\\ \hline \dfrac{a}{L_4} & \dfrac{1-a}{L_4} & -\dfrac{aR_5}{L_4}\end{array}\right],$$

which agrees with the results previously obtained (see equation 5.55).

To find the **b** matrix we will calculate equation 5.79. Since only the $\mathbf{v}_{sR}$ vector is present we need only two elements of **b**:

$$\mathbf{b}_{14}^{1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} aG_3 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix} = \begin{bmatrix} aG_3 & 1/R_6 & 0 \\ -aG_3 & 0 & 1/R_7 \end{bmatrix}$$

$$\mathbf{b}_{24}^{1} = -[1][aR_5][1 \quad 0 \quad 0] \begin{bmatrix} 1/R_5 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix} = -[a \quad 0 \quad 0]$$

Therefore, the reduced **b** matrix is

$$\mathbf{b} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_{14}^{1} \\ \mathbf{b}_{24}^{1} \end{bmatrix} = \begin{bmatrix} aG_3/C_1 & 1/R_6C_1 & 0 \\ -aG_3/C_2 & 0 & 1/R_7C_2 \\ -a/L_4 & 0 & 0 \end{bmatrix}$$

which also agrees with the results in equation 5.55. Note that a voltage source in link 5 is absent ($v_{sR5} = 0$), therefore the above matrix can be reduced even more, namely

$$\mathbf{b} = \begin{bmatrix} 1/R_6C_1 & 0 \\ 0 & 1/R_7C_2 \\ 0 & 0 \end{bmatrix}$$

which is exactly the same as in equation 5.55.

Comparing the systematic method for writing state equations with the intuitive approach, which we first presented in the previous sections, we may conclude that it is rather complicated. In many practical instances, the final results can be arrived at much easier and faster by following the intuitive approach. However, the systematic method has an appreciable advantage for computer-aided analysis, since it can be easily programmed.

5.7 COMPLETE SOLUTION OF THE STATE MATRIX EQUATION

We will now turn to the solution of the state equation of the form of equation 5.44a, repeated here for convenience:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{w}(t). \tag{5.80}$$

5.7.1 The natural solution

We will begin by considering the natural or zero-input (non-forced) solution; that is $\mathbf{w}(t) = 0$. Equation 5.80 then simplifies to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \text{or} \quad \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}. \tag{5.81}$$

It is customary to compare a vector problem with its scalar version. In this case, the scalar version of equation 5.81 is

$$\frac{dx(t)}{dt} = ax(t). \tag{5.82}$$

The solution of equation 5.82, that satisfies the initial condition $x(0)$, is

$$x(t) = e^{at}x(0).$$

Suppose we try the same form for the solution of equation 5.81, that is

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{x}(0). \tag{5.83}$$

where $\mathbf{e}^{\mathbf{A}t}$ is called the *matrix exponential* and is an example of a function of matrix $\mathbf{A}$.

5.7.2 Matrix exponential

In mathematics the matrix exponential is defined similarly to a scalar exponential (or complex exponential), i.e. in terms of the power series expansion:

$$\mathbf{e}^{\mathbf{A}t} = \mathbf{1} + \frac{t}{1!}\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots + \frac{t^k}{k!}\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{A}^k. \tag{5.84}$$

Since $\mathbf{A}$ is a square matrix of order n, the matrix exponential $\mathbf{e}^{\mathbf{A}t}$ is also a square matrix of order n.

Example 5.2

As an example, let us take the matrix of Example 5.1, namely

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

then

$$\mathbf{A}^2 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

and

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \frac{t^3}{6}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - t + \dfrac{t^3}{3} + \cdots & -t + t^2 - \dfrac{t^3}{3} + \cdots \\ t - t^2 + \dfrac{t^3}{3} + \cdots & 1 - t + \dfrac{t^3}{3} + \cdots \end{bmatrix}. \tag{5.85}$$

As can be seen from equation 5.85, each of the elements of the matrix $e^{\mathbf{A}t}$ is a continuous function of t. Term-by-term differentiation of the matrix exponential (equation 5.84) results in

$$\frac{d}{dt}(\mathbf{e}^{\mathbf{A}t}) = \mathbf{A} + t\mathbf{A}^2 + \frac{t^2}{2!}\mathbf{A}^3 + \frac{t^3}{3!}\mathbf{A}^4 + \cdots$$
$$= \mathbf{A}\left(\mathbf{1} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots\right) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}, \qquad (5.86)$$

i.e., the formula for the derivative of a matrix exponential is the same as it is for a scalar exponential. Substituting equation 5.83 into the matrix differential equation 5.81, results in identity:

$$\mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{x}(0).$$

Thus, we have established that equation 5.83 is indeed the solution to equation 5.81.

We must now show that the inverse of a matrix exponential exists and equals $(\mathbf{e}^{\mathbf{A}t})^{-1} = \mathbf{e}^{-\mathbf{A}t}$. For the latter we can write

$$\mathbf{e}^{-\mathbf{A}t} = \mathbf{1} - \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} - \mathbf{A}^3\frac{t^3}{3!} + \cdots + (-1)^k\mathbf{A}^k\frac{t^k}{k!} + \cdots.$$

Now let this series be multiplied by the series for the positive exponential in equation 5.84. This term-by-term multiplication results in **1** since all other terms are cancelled. Thus,

$$\mathbf{e}^{\mathbf{A}t}\mathbf{e}^{-\mathbf{A}t} = \mathbf{1}.$$

This result tells us that the matrix $\mathbf{e}^{-\mathbf{A}t}$ is the inverse of $\mathbf{e}^{\mathbf{A}t}$, since by definition the product of the matrix by its inverse gives a unit matrix. This result can be used, first of all, to show that in general if the initial vector $\mathbf{x}(0)$ is known for some time, for instance t_0, namely $\mathbf{x}_{nat}(t_0)$ then the solution will be

$$\mathbf{x}_n(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0). \qquad (5.87)$$

Indeed, substituting $t = t_0$, results in identity:

$$\mathbf{x}_n(t_0) = \mathbf{e}^{\mathbf{A}t_0}\mathbf{e}^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \mathbf{1}\mathbf{x}(t_0),$$

where we have used

$$\mathbf{e}^{\mathbf{A}+\mathbf{B}} = \mathbf{e}^{\mathbf{A}}\cdot\mathbf{e}^{\mathbf{B}}.$$

(This can be verified by using equation 5.84 for both sides of equality.)

5.7.3 The particular solution

To find the complete solution to equation 5.80, we must now find the particular solution to the differential equation, i.e. the forced response. For this purpose, assume a solution of the form

$$\mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t), \qquad (5.88)$$

where $\mathbf{q}(t)$ is an unknown function to be determined. In order to be a solution, equation 5.88 has to satisfy the differential equation. Substituting equation 5.88 in equation 5.80 gives

$$\frac{d}{dt}[\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t)] = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{b}\mathbf{w}(t),$$

or

$$\mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{e}^{\mathbf{A}t}\frac{d\mathbf{q}(t)}{dt} = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{b}\mathbf{w}(t).$$

Thus

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{e}^{-\mathbf{A}t}\mathbf{b}\mathbf{w}(t). \tag{5.89}$$

Integrating, we obtain

$$\mathbf{q}(t) = \mathbf{q}(t_0) + \int_{t_0}^{t} \mathbf{e}^{-\mathbf{A}\tau}\mathbf{b}\mathbf{w}(\tau)d\tau.$$

Thus, the particular solution is

$$\mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau.$$

To evaluate $\mathbf{q}(t_0)$, we use the complete solution being evaluated at t_0

$$\mathbf{x}(t)|_{t=t_0} = \mathbf{x}_n(t) + \mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau \bigg|_{t=t_0},$$

or

$$\mathbf{x}(t_0) = \mathbf{x}(t_0) + \mathbf{e}^{\mathbf{A}t_0}\mathbf{q}(t_0) + \mathbf{0},$$

which implies that $\mathbf{q}(t_0) = \mathbf{0}$.

Hence, finally the complete solution of the state equation 5.80 is

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau. \tag{5.90}$$

To evaluate this solution the basic calculation is a determination of the matrix exponential $\mathbf{e}^{\mathbf{A}t}$. This will be discussed in the next subsection.

5.8 BASIC CONSIDERATIONS IN DETERMINING FUNCTIONS OF A MATRIX

In this section, we shall examine two methods of computing $\mathbf{e}^{\mathbf{A}t}$ in closed form. This matrix exponential is a particular function of a matrix. The simplest functions of a matrix are powers of a matrix and polynomials. As we have seen, the matrix exponential can be represented by an infinite series of such functions. The matrix polynomial has the form

$$f(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{1}. \tag{5.91}$$

The generalization of polynomials is an infinite series:

$$f(\mathbf{A}) = a_0\mathbf{1} + a_0\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_k\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty} a_k\mathbf{A}^k. \tag{5.92}$$

The function $f(\mathbf{A})$ is itself a matrix, and in the last case each of the matrix elements is an infinite series. This matrix series is said to converge if each of the element series converges.

We will begin with a brief description of some of the properties of matrixes that will be useful in our studies.

5.8.1 Characteristic equation and eigenvalues

An algebraic equation that often appears in network transient analysis is

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{5.93}$$

where $\mathbf{A}$ is a square matrix of order n. The problem is to find scalars λ and vectors $\mathbf{x}$ that satisfy this equation. A value of λ for which a nontrivial solution of $\mathbf{x}$ exists, is called an eigenvalue, or characteristic value of $\mathbf{A}$. The corresponding vector $\mathbf{x}$ is called an eigenvector, or characteristic vector, of $\mathbf{A}$. After collecting the terms on the left-hand side, we have

$$[\lambda\mathbf{1} - \mathbf{A}]\mathbf{x} = \mathbf{0}. \tag{5.94}$$

This equation will have a nontrivial solution for $\mathbf{x}$ only if the matrix $[\lambda\mathbf{1} - \mathbf{A}]$ is singular, i.e.,

$$\det[\lambda\mathbf{1} - \mathbf{A}] = \mathbf{0}. \tag{5.95}$$

This equation is known as the characteristic equation associated with $\mathbf{A}$. It is also closely related to the auxiliary (characteristic) equation of the corresponding differential equation of order n for the system. The determinant on the left-hand side of equation 5.95 is actually a polynomial of degree n in λ and is called the characteristic polynomial of $\mathbf{A}$. For each value of λ that satisfies the characteristic equation, a nontrivial solution of equation 5.94 can be found. To illustrate this procedure, consider the following example.

Example 5.3

Let us find the eigenvalues and eigenvectors of a matrix of the second order

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

The characteristic polynomial is also of order two:

$$\det\left\{\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\right\}=\det\begin{bmatrix}\lambda-2 & -1\\ -3 & \lambda-4\end{bmatrix}=\lambda^2-6\lambda+5$$

$$=(\lambda-5)(\lambda-1)=g(\lambda).$$

Thus, $\lambda^2-6\lambda+5=0$ is the characteristic equation of the matrix. The roots of the characteristic equation, or the eigenvalues, are

$$\lambda_1=5 \quad \text{and} \quad \lambda_2=1.$$

To obtain the eigenvector corresponding to the eigenvalue $\lambda_1=5$, we solve equation 5.94 by using the given matrix **A**. Thus

$$\left\{\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\right\}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

or

$$\begin{bmatrix}3 & -1\\ -3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{and} \quad x_2=3x_1.$$

Therefore

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 3x_1\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}[x_1] \quad \text{for any value of } x_1.$$

The eigenvector corresponding to the eigenvalue $\lambda_2=1$ is obtained similarly.

$$\begin{bmatrix}-1 & -1\\ -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

from which

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}[x_1] \quad \text{for any value of } x_1.$$

The first method to be discussed for finding functions of a matrix is based on the Caley-Hamilton theorem.

5.8.2 The Caley-Hamilton theorem

This theorem states that *every square matrix satisfies its own characteristic equation*. For example, if we substitute **A** for λ in the characteristic equation of Example 5, we obtain the matrix equation

$$g(\mathbf{A})=\mathbf{A}^2-6\mathbf{A}+5\cdot\mathbf{1}=\mathbf{0},$$

where, again, **1** is an identity matrix and **0** is a matrix whose elements are all zero. Thus,

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - 6\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 18 & 19 \end{bmatrix} - \begin{bmatrix} 12 & 6 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The equation is certainly satisfied in this example.

The Caley-Hamilton theorem permits us to reduce the order of a matrix polynomial of any higher order to be of an order no greater than $n-1$, where n is the order of the matrix. For example, if $\mathbf{A}$ is a square matrix of order 3, then its characteristic equation is

$$g(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{5.96}$$

and by the Caley-Hamilton theorem we have

$$\mathbf{A}^3 + a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{1} = \mathbf{0}.$$

Then

$$\mathbf{A}^3 = -a_2\mathbf{A}^2 - a_1\mathbf{A} - a_0\mathbf{1}. \tag{5.97}$$

Thus, $\mathbf{A}^3$ may be expressed in terms of the matrixes of an order not higher than 2 and identity matrix. Hence, the given polynomial of order 3 is reduced to a polynomial of order 2. To extend these results to polynomials of an even higher order, we multiply equation 5.97 throughout by $\mathbf{A}$ to obtain

$$\mathbf{A}^4 = -a_2\mathbf{A}^3 - a_1\mathbf{A}^2 - a_0\mathbf{A}. \tag{5.98}$$

Substituting equation 5.97 for $\mathbf{A}^4$, we obtain

$$\mathbf{A}^4 = (a_2^2 - a_1)\mathbf{A}^2 + (a_2a_1 - a_0)\mathbf{A} + a_2a_0\mathbf{1}. \tag{5.99a}$$

To generalize these results, let us develop an iterative formula for expressing higher powers of $\mathbf{A}$. We assign the obtained coefficients in equation 5.99 by upper script, as follows

$$\mathbf{A}^4 = a_2^{(1)}\mathbf{A}^2 + a_1^{(1)}\mathbf{A} + a_0^{(1)}\mathbf{1}. \tag{5.99b}$$

Multiplying this expression throughout by $\mathbf{A}$, and collecting like terms, yields

$$\mathbf{A}^5 = (-a_2a_2^{(1)} + a_1^{(1)})\mathbf{A}^2 + (-a_1a_2^{(1)} + a_0^{(1)})\mathbf{A} + (-a_0a_2^{(1)})\mathbf{1} = a_2^{(2)}\mathbf{A}^2 + a_1^{(2)}\mathbf{A} + a_0^{(2)}\mathbf{1},$$

where again $a_2^{(2)}, a_1^{(2)}, a_0^{(2)}$ are the new coefficients and a_2, a_1, a_0 are as before the coefficients of the characteristic equation 5.96. Now the iterative formula for this case, $n = 3$, can be written as

$$\begin{aligned}\mathbf{A}^{3+k} &= (-a_2a_2^{(k-1)} + a_1^{(k-1)})\mathbf{A}^2 + (-a_1a_2^{(k-1)} + a_0^{(k-1)})\mathbf{A} + (-a_0a_2^{(k-1)})\mathbf{1} \\ &= a_2^{(k)}\mathbf{A}^2 + a_1^{(k)}\mathbf{A} + a_0^{(k)}1. \end{aligned} \tag{5.100}$$

Note that this formula also works fine for the first calculation of $\mathbf{A}^4$ (equation

5.99) if the coefficients in equation 5.97 are assigned as $a_2^{(0)} = -a_2$, $a_1^{(0)} = -a_1$ and $a_0^{(0)} = -a_0$. Generalizing this result (equation 5.100) for any matrix of order n, we can write

$$\mathbf{A}^{n+k} = (-a_{n-1}a_{n-1}^{(k-1)} + a_{n-2}^{(k-1)})\mathbf{A}^{n-1} + (-a_{n-2}a_{n-1}^{(k-1)} + a_{n-3}^{(k-1)})\mathbf{A}^{n-2} + \cdots + (-a_0 a_{n-1}^{(k-1)})\mathbf{1}. \tag{5.101}$$

This gives us an expression for $\mathbf{A}^{n+k}$, $k = 0, 1, 2, \ldots$, in terms of $\mathbf{A}^{n-1}$, $\mathbf{A}^{n-2}, \ldots, \mathbf{A}$ and $\mathbf{1}$.

Continuing this process, we see that any power of $\mathbf{A}$ can be represented as a weighted polynomial in $\mathbf{A}$ of an order, at most $n-1$. Hence, functions of matrixes, including $\mathbf{e}^{\mathbf{A}t}$, that can be expressed as a polynomial(*)

$$f(\mathbf{A}) = \alpha_0\mathbf{1} + \alpha_1\mathbf{A} + \cdots + \alpha_k\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty} \alpha_k\mathbf{A}^k, \tag{5.102}$$

may be reduced to the expression

$$f(\mathbf{A}) = \beta_0\mathbf{1} + \beta_1\mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1} = \sum_{k=0}^{n-1} \beta_k\mathbf{A}^k. \tag{5.103}$$

Here, the coefficients $\beta_0, \beta_1, \ldots, \beta_{n-1}$ are functions of $a_0, a_1, \ldots, a_{n-1}$ and $\alpha_0, \alpha_1, \ldots$. Their approximate calculation can be carried out by the iterative method used in the calculation of higher powers of $\mathbf{A}$ in equation 5.101 and by using equation 5.102. However this straightforward method can be lengthy.

Example 5.4

(a) Let us first calculate a simple matrix function $f(\mathbf{A}) = \mathbf{A}^4$, where $\mathbf{A}$ is the matrix of the previous example. Since the characteristic equation of $\mathbf{A}$ is $\lambda^2 - 6\lambda + 5 = 0$, we have

$$\mathbf{A}^2 = 6\mathbf{A} - 5\cdot\mathbf{1},$$

where $a_1 = -6$ and $a_0 = 5$. Using an iterative formula, and noting that in the first calculation $a_1^{(0)} = -a_1$ and $a_0^{(0)} = -a_0$, yields

$$\begin{aligned}\mathbf{A}^3 &= [-a_1 a_1^{(0)} + a_0^{(0)}]\mathbf{A} + (-a_0 a_1^{(0)})\mathbf{1}\\ &= [6\cdot 6 - 5]\mathbf{A} + (-5\cdot 6)\mathbf{1} = 31\mathbf{A} - 30\,\mathbf{1},\end{aligned}$$

where $a_1^{(1)} = 31$ and $a_0^{(1)} = -30$. Hence,

$$\mathbf{A}^4 = [(6)(31) - 30]\mathbf{A} - 5\cdot 31\,\mathbf{1} = 156\mathbf{A} - 155\,\mathbf{1},$$

and finally

$$\mathbf{A}^4 = 156\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 155 & 0 \\ 0 & 155 \end{bmatrix} = \begin{bmatrix} 157 & 156 \\ 468 & 469 \end{bmatrix}$$

(*)In general, any analytic function of matrix $\mathbf{A}$ can be expressed as a polynomial in $\mathbf{A}$ of an order no greater than one less than the order of $\mathbf{A}$. For proof see N. Balabanian and T. A. Bickart (1969) *Electrical Network Theory*, John Wiley & Sons.

(b) As a second example, let us calculate a matrix potential $f(\mathbf{A}) = \mathbf{e}^{\mathbf{A}t}$ for $t =$ 1 s, using the approximation up to fifth term:

$$\begin{aligned}\mathbf{e}^{\mathbf{A}} &\cong \mathbf{1} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 \\ &= \mathbf{1} + \mathbf{A} + \frac{1}{2}(-5\cdot\mathbf{1} + 6\mathbf{A}) + \frac{1}{6}(-30\cdot\mathbf{1} + 31\mathbf{A}) + \frac{1}{24}(155\cdot\mathbf{1} + 156\mathbf{A}) \\ &= -12.96\cdot\mathbf{1} + 15.67\mathbf{A}\end{aligned}$$

and finally

$$\mathbf{e}^{\mathbf{A}} \cong \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix} + 15.7 \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 18.4 & 15.7 \\ 47.1 & 49.8 \end{bmatrix}.$$

We shall next develop an easier, one-step method for finding β-coefficients in the function of matrix expression (equation 5.103). Let us return to the characteristic equation of matrix $\mathbf{A}$

$$g(\lambda) = |\lambda\mathbf{1} - \mathbf{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0. \tag{5.104}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are the roots of the characteristic equation 5.104, obviously satisfy the equation 5.104 as well as matrix $\mathbf{A}$ (in accordance with the Caleg-Hamilton theorem). Therefore, using the same procedure as before, we can derive an expression similar to equation 5.103 for the eigenvalues instead of the matrix by itself, namely:

$$f(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1} = \sum_{k=0}^{n-1} \beta_k\lambda^k. \tag{5.105}$$

It is understandable that this expression holds for any λ that is a solution of the characteristic equation 5.104, that is for any eigenvalue of the matrix $\mathbf{A}$.

(a) *Distinct eigenvalues*

Assume first that the eigenvalues are *distinct*; that is, that none is repeated. Substituting $\lambda_1, \lambda_2, \ldots, \lambda_n$ in equation 5.105 gives n equations in n unknown β's:

$$\begin{aligned} &\beta_0 + \beta_1\lambda_1 + \beta_1\lambda_1^2 + \cdots + \beta_{n-1}\lambda_1^{n-1} = f(\lambda_1) \\ &\beta_0 + \beta_1\lambda_1 + \beta_2\lambda_2^2 + \cdots + \beta_{n-1}\lambda_2^{n-1} = f(\lambda_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\beta_0 + \beta_1\lambda_n + \beta_2\lambda_n^2 + \cdots + \beta_{n-1}\lambda_1^{n-1} = f(\lambda_n). \end{aligned} \tag{5.106}$$

The coefficients $\beta_0, \beta_1, \ldots, \beta_{n-1}$ can then be obtained as the solution to this linear system of scalar equations, i.e. the inversion of the set of equations 5.106 gives the solution. With the known β-coefficients, the function of the matrix representation problem is solved:

$$f(\mathbf{A}) = \sum_{k=0}^{n-1} \beta_k \mathbf{A}^k. \tag{5.107}$$

Example 5.5

Let us illustrate this process with the same simple example (as in Example 5.4):

(a) Find $f(\mathbf{A}) = \mathbf{A}^4$, if $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

The characteristic equation is (see Example 5.3)

$$g(\lambda) = \lambda^2 - 6\lambda + 5 = 0.$$

Thus, the eigenvalues are

$$\lambda_1 = 5, \quad \lambda_2 = 1.$$

In accordance with equation 5.106, we have

$$\beta_0 + \beta_1 5 = 5^4,$$

$$\beta_0 + \beta_1 1 = 1^4.$$

Solving these simple equations for unknowns β_0 and β_1, gives

$$\beta_1 = 156, \quad \beta_0 = -155.$$

The solution for $\mathbf{A}^4$ is found by using equation 5.107

$$f(\mathbf{A}) = \mathbf{A}^4 = -155 \cdot \mathbf{1} + 156 \cdot \mathbf{A}$$

which is the same as the results obtained in the previous example.

(b) Find $f(\mathbf{A}) = \mathbf{e}^{\mathbf{A}t}$ for the same matrix $\mathbf{A}$

The equations for unknowns β_0 and β_1 in this case will be

$$\beta_0 + 5\beta_1 = e^{5t},$$

$$\beta_0 + \beta_1 = e^{t}.$$

Solving this equation gives

$$\beta_1 = \tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^{t}, \quad \beta_0 = -\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^{t}.$$

Thus, the matrix exponential is

$$\begin{aligned}\mathbf{e}^{\mathbf{A}t} &= (-\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^{t})\mathbf{1} + (\tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^{t})\mathbf{A} \\ &= (-\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^{t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^{t})\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.\end{aligned}$$

By an obvious rearrangement, this becomes

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{4}e^{5t} + \frac{3}{4}e^{t} & \frac{1}{4}e^{5t} - \frac{1}{4}e^{t} \\ \frac{3}{4}e^{5t} - \frac{3}{4}e^{t} & \frac{3}{4}e^{5t} + \frac{1}{4}e^{t} \end{bmatrix}. \tag{5.108}$$

It is interesting to compare these results with those obtained in the previous example. The approximate, up to fifth term, evaluation of the exponents e^5 and e^1 ($t = 1$ s) gives

$$e^5 \cong 1 + 5 + \frac{1}{2!}5^2 + \frac{1}{3!}5^3 + \frac{1}{4!}5^4 = 65.4$$

$$e^1 \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.71.$$

Substituting these results in equation 5.108 yields

$$\mathbf{e}^{\mathbf{A}} \cong \begin{bmatrix} 18.4 & 15.6 \\ 47.0 & 49.7 \end{bmatrix}$$

which agrees with the previous results.

Therefore, the series form of the exponential may permit some approximate numerical results; it does not lead to a closed form. However, with the help of the Caley-Hamilton theorem, we obtained the closed-form equivalent for the exponential $\mathbf{e}^{\mathbf{A}t}$ (equation 5.107). We shall now return our consideration to the complete solution of the state equation in the form of equation 5.90, repeated here for convenience:

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau. \tag{5.109}$$

The following example illustrates this computation.

Example 5.6

Find the complete solution of the state equation describing the circuit in Fig. 5.9, considered before. For the sake of convenience, it is redrawn here again in Fig. 5.11(a). Let the circuit element values be $C_1 = 1$ F, $C_2 = 2$ F, $L_4 = 1$ H, $G_3 = 1$ S, $R_5 = 1\ \Omega$, $R_6 = 2/7\ \Omega$, $R_7 = 1/3\ \Omega$.

Solution

Substituting these parameters into equation 5.55, we obtain the following **A** matrix

$$\mathbf{A} = \begin{bmatrix} -4 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \tag{5.110}$$

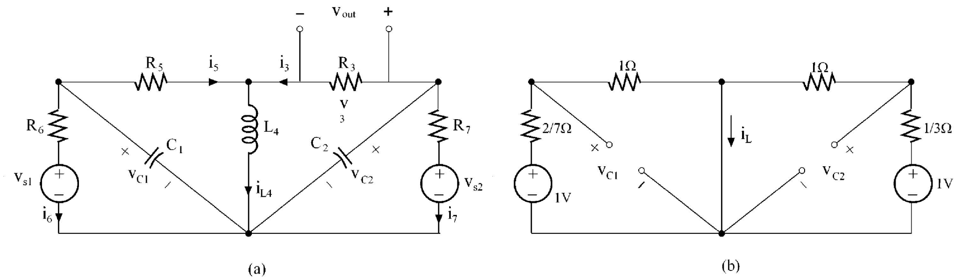

Figure 5.11 A circuit of Example 5.6 (a) and its steady-state equivalent (b).

The characteristic equation is

$$g(\lambda) = |\lambda \cdot \mathbf{1} - \mathbf{A}| = \begin{vmatrix} \lambda + 4 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \lambda + \frac{7}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda + \frac{1}{2} \end{vmatrix} = 0.$$

Thus,

$$g(\lambda) = (\lambda + 4)[(\lambda + \tfrac{7}{4})(\lambda + \tfrac{1}{2}) + \tfrac{1}{4}] = 0.$$

Simplifying yields

$$(\lambda + 4)(\lambda^2 + \tfrac{9}{4}\lambda + \tfrac{9}{8}) = 0. \tag{5.111}$$

Thus, the eigenvalues of **A** are

$$\lambda_{1,2} = -\frac{9}{8} \pm \sqrt{\left(\frac{9^2}{8^2} - \frac{9}{8}\right)} = -1.125 \pm 0.375$$

or

$$\lambda_1 = -0.75,\ \lambda_2 = -1.5,\ \lambda_3 = -4.$$

Using the results of equation 5.106, we can evaluate β_0, β_1, and β_2 from the equations

$$\beta_0 - 0.75\beta_1 + (-0.75)^2\beta_2 = e^{-0.75t}$$
$$\beta_0 - 1.5\beta_1 + (-1.5)^2\beta_2 = e^{-1.5t}$$
$$\beta_0 - 4\beta_1 + (-4)^2\beta_2 = e^{-4t},$$

which in the matrix form are

$$\begin{bmatrix} 1 & -0.75 & 0.5625 \\ 1 & -1.5 & 2.25 \\ 1 & -4 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} e^{-0.75t} \\ e^{-1.5t} \\ e^{-4t} \end{bmatrix}. \tag{5.113}$$

The solution for β's is found by inversion, as

$$\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}=\begin{bmatrix}1 & -0.75 & 0.5625\\ 1 & -1.5 & 2.25\\ 1 & -4 & 16\end{bmatrix}^{-1}\begin{bmatrix}e^{-0.75t}\\ e^{-1.5t}\\ e^{-4t}\end{bmatrix}$$

$$=\begin{bmatrix}2.462 & -1.6 & 0.1385\\ 2.256 & -2.533 & 0.2769\\ 0.4103 & -0.5333 & 0.1231\end{bmatrix}\begin{bmatrix}e^{-0.75t}\\ e^{-1.5t}\\ e^{-4t}\end{bmatrix}$$

$$=\begin{bmatrix}2.462e^{-0.75t} & -1.6e^{-1.5t} & 0.1385e^{-4t}\\ 2.256e^{-0.75t} & -2.533e^{-1.5t} & 0.2769e^{-4t}\\ 0.4103e^{-0.75t} & -0.5333e^{-1.5t} & 0.1231e^{-4t}\end{bmatrix}. \qquad (5.114)$$

With β's now known, matrix $\mathbf{e}^{\mathbf{A}t}$ will be

$$\mathbf{e}^{\mathbf{A}t}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\beta_0+\begin{bmatrix}-4 & 0.5 & -0.5\\ 0.25 & -1.75 & -0.25\\ 0.5 & 0.5 & -0.5\end{bmatrix}\beta_1$$

$$+\begin{bmatrix}15.87 & -3.125 & 2.125\\ -1.563 & 3.063 & 0.438\\ -2.125 & -0.875 & -0.125\end{bmatrix}\beta_2.$$

Substituting equation 5.114 for β's and collecting like terms yields the final results

$$\mathbf{e}^{\mathbf{A}t}=\begin{bmatrix}-0.048 & -0.154 & -0.256\\ -0.077 & -0.229 & -0.384\\ 0.256 & 0.769 & 1.283\end{bmatrix}e^{-0.75t}+\begin{bmatrix}0.066 & 0.4 & 0.133\\ 0.2 & 1.2 & 0.4\\ -0.133 & -0.8 & -0.267\end{bmatrix}e^{-1.5t}$$

$$+\begin{bmatrix}0.985 & -0.246 & 0.123\\ -0.123 & 0.031 & -0.015\\ -0.123 & 0.031 & -0.015\end{bmatrix}e^{-4t}. \qquad (5.115)$$

Now suppose that the initial state vector at $t_0=0$ is $\mathbf{x}(0)=[0.5\ \ 1.5\ \ 1]^T$, then the natural solution (for $\mathbf{w}(t)=\mathbf{0}$) in equation 5.109 is

$$\mathbf{x}_{nat}(t)=\mathbf{e}^{\mathbf{A}t}\mathbf{x}(0)=\begin{bmatrix}-0.511e^{-0.75t} & +0.767e^{-1.5t} & +0.246e^{-4t}\\ -0.766e^{-0.75t} & +2.30e^{-1.5t} & -0.031e^{-4t}\\ 2.564e^{-0.75t} & -1.534e^{-1.5t} & -0.031e^{-4t}\end{bmatrix}.$$

(5.116)

The next step is to find the particular or forced solution of the state equation. Let the input vector $\mathbf{w}(t) = [1 \;\; 1]^T$. Substituting the circuit parameters into matrix $\mathbf{b}$ in equation 5.55, we obtain

$$\mathbf{b} = \begin{bmatrix} 3.5 & 0 \\ 0 & 1.5 \\ 0 & 0 \end{bmatrix}. \tag{5.117}$$

Since the input is a constant (d.c.), evaluating the integral in equation 5.55 results, for $t_0 = 0$, in

$$\int_0^t \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{bw}\, d\tau = -\mathbf{A}^{-1}\mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{bw}|_0^t = \mathbf{A}^{-1}[\mathbf{e}^{\mathbf{A}t} - \mathbf{1}]\mathbf{bw}, \tag{5.118}$$

where the inverse of the $\mathbf{A}$ matrix is found as follows

$$\mathbf{A}^{-1} = \begin{bmatrix} -4 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} -0.222 & 0 & 0.222 \\ 0 & -0.5 & 0.25 \\ -0.222 & -0.5 & -1.528 \end{bmatrix}. \tag{5.119}$$

Performing now, all the calculations in equation 5.118, with equations 5.119, 5.115, 5.117 and $\mathbf{w} = [1 \;\; 1]^T$, we obtain the particular solution

$$\mathbf{x}_{par}(t) = \begin{bmatrix} 0.547e^{-0.75t} - 0.556e^{-1.5t} - 0.769e^{-4t} + 0.778 \\ 0.821e^{-0.75t} - 1.667e^{-1.5t} + 0.096e^{-4t} + 0.750 \\ -2.735e^{-0.75t} + 1.111e^{-1.5t} + 0.096e^{-1.5t} + 1.528 \end{bmatrix}. \tag{5.120}$$

The final result of the complete solution is simply obtained by combining the above two solutions: the natural (equation 5.116) and the particular (equation 5.120), which leads to

$$\mathbf{x}(t) = \mathbf{x}_{nat} + \mathbf{x}_{par} = \begin{bmatrix} 0.034e^{-0.75t} + 0.211e^{-1.5t} - 0.523e^{-4t} + 0.778 \\ 0.052e^{-0.75t} + 0.633e^{-1.5t} + 0.065e^{-4t} + 0.750 \\ -0.171e^{-0.75t} - 0.423e^{-1.5t} + 0.065e^{-4t} + 1.528 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}. \tag{5.121}$$

Figure 5.12 shows the state variables v_{c1}, v_{c2}, i_{L4} behavior versus time.

The computer calculation of the state variables in the above example, using the MATHCAD program is shown in Appendix I. (Note that the computing results are slightly different from those obtained above.)

To complete this example, suppose that voltage v_3 is of interest. Then the

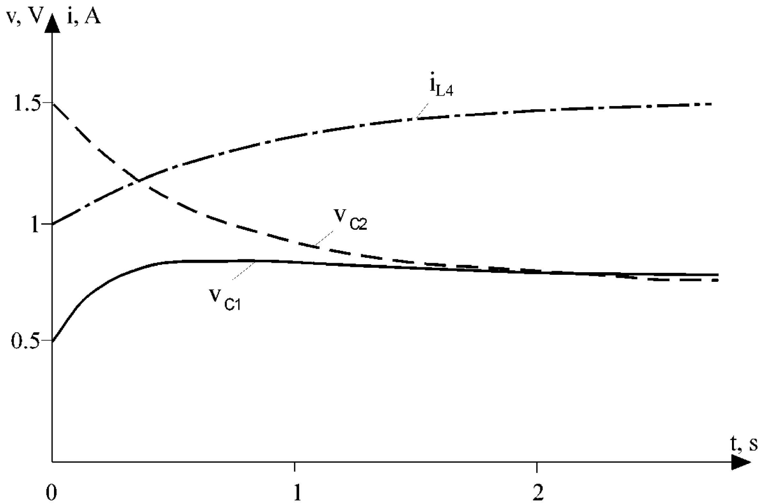

Figure 5.12 Two capacitor voltages and inductor current curves versus time of Example 5.6.

output equation 5.56 simplifies to

$$v_3(t) = [-a \quad a \quad aR_5]\mathbf{x}(t) = [-\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}]\begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}.$$

Thus, the output voltage is

$$\begin{aligned} v_{out}(t) &= v_3 = \tfrac{1}{2}(-v_{c1} + v_{c2} + i_{L4}) \\ &= -0.077e^{-0.75t} - 0.0005e^{-1.5t} + 0.327e^{-4t} + 0.750\ \text{V}. \end{aligned} \tag{5.122}$$

Note that by inspection of the given circuit in its d.c. steady-state behavior, i.e. the capacitors are open-circuited and the inductor is short-circuited as shown in Fig. 5.11(b), we may find

$$v_{c1}(\infty) = \frac{v_{s1}}{R_5 + R_6} R_5 = \frac{1}{1 + 2/7} \cdot 1 = 0.778\ \text{V}$$

$$v_{c2}(\infty) = \frac{v_{s2}}{R_3 + R_7} R_3 = \frac{1}{1 + 1/3} \cdot 1 = 0.75\ \text{V}$$

$$i_L(\infty) = v_{c1}/R_5 + v_{c2}/R_3 = 0.778 + 0.75 = 1.528\ \text{A},$$

which is in agreement with the final results in equation 5.121.

(b) *Multiple eigenvalues*

If some of the eigenvalues of **A** (roots of the characteristic equation $g(\lambda) \neq 0$)

are not distinct and there are repeated values (for example $\lambda_1 = \lambda_2$), then in this case, the number of independent equations in 5.106 would be fewer than n unknown coefficients β. The following theorem allows us to extend the solution for finding all β's to the case of repeated eigenvalues.

Theorem:(*) Let **A** be the $n \times n$ matrix with n_0 distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n0}$ and m multiple eigenvalues ($n_0 < n$, if no eigenvalue is repeated, then $n_0 = n$). Let the eigenvalue λ_i occur with multiplicity r_i, and define the polynomials

$$\mathbf{P}(\mathbf{A}) = \sum_{k=0}^{n-1} \beta_k \mathbf{A}, \tag{5.123}$$

and

$$P(\lambda) = \sum_{k=0}^{n-1} \beta_k \lambda^k. \tag{5.124}$$

Then the matrix function $f(\mathbf{A})$ is identical to the matrix polynomial $\mathbf{P}(\mathbf{A})$ (see 5.107) if the following conditions are obeyed:

for each distinct eigenvalue

$$f(\lambda_i) = P(\lambda_i) \quad i = 1, 2, \ldots, n_0 \tag{5.125a}$$

for each multiple eigenvalue

$$\frac{d^q}{d\lambda^q} f(\lambda)|_{\lambda=\lambda_i} = \frac{d^q}{d\lambda^q} P(\lambda)|_{\lambda=\lambda_i},$$

$$i = n_{0+1}, n_{0+2}, \ldots, n_{0+m}, \quad q = 0, 1, 2, \ldots, r_i - 1 \tag{5.125b}$$

that the first condition (equation 5.125a) gives us only n_0 ($n_0 < n$) independent equations for finding n unknown β-coefficients. However, the second condition (equation 5.125b) yields the remaining equations needed to solve for $\beta_0, \beta_1, \ldots, \beta_{n-1}$. For this purpose equation 5.125b shall be rewritten in terms of the unknown β's

$$\frac{d^q}{d\lambda^q} f(\lambda)|_{\lambda=\lambda_i} = \frac{d^q}{d\lambda^q} \sum_{k=0}^{n-1} \beta_k \lambda^k|_{\lambda=\lambda_i} = \sum_{k=q}^{k-1} k(k-1)\cdots(k-q+1)\beta_k \lambda_i^{k-q},$$

$$i = n_{0+1}, n_{0+2}, \ldots, n_{0+m}, \quad q = 0, 1, 2, \ldots, r - 1 \tag{5.126}$$

The total number of independent equations, therefore, will be

$$n_0 + \sum_1^m r_i = n.$$

Example 5.7

As an example of the determination of a matrix function when **A** has multiple

(*)The proof can be found in the book by Balabanian N. and Bickart T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

eigenvalues, let us consider the same circuit in Fig. 5.11 of the previous example with slightly different parameters, namely: $R_6 = 1/3\,\Omega$, $R_7 = 2/5\,\Omega$ (the rest of the parameters are the same). Suppose we wish to find $\mathbf{e}^{\mathbf{A}t}$.

Solution

The $\mathbf{A}$ matrix in this case will be

$$\mathbf{A} = \begin{bmatrix} -\frac{7}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{3}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

which yields the characteristic equation

$$g(\lambda) = \begin{bmatrix} \lambda + \frac{7}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \lambda + \frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda + \frac{1}{2} \end{bmatrix}$$
$$= (\lambda + \tfrac{7}{2})(\lambda + \tfrac{3}{2})(\lambda + \tfrac{1}{2}) + \tfrac{1}{4}\lambda + \tfrac{3}{8} = (\lambda + \tfrac{7}{2})(\lambda^2 + 2\lambda + 1) = 0.$$

Thus, the eigenvalues are $\lambda_1 = -\frac{7}{2}$ and double $\lambda_2 = -1$, i.e. the multiplicity $r = 2$. Therefore, for the first distinct eigenvalue, in accordance with equation 5.125a, we have

$$\beta_0 + \beta_1(-\tfrac{7}{2}) + \beta_2(-\tfrac{7}{2})^2 = e^{-(7/2)t},$$

and for the double eigenvalue, in accordance with equation 5.125b we have

$$\beta_0 + \beta_1(-1) + \beta_2(-1)^2 = e^{-t}, \quad q = 0$$
$$\beta_1 + 2\beta_2(-1) = te^{-t}, \quad q = 1.$$

Since

$$\left.\frac{df(\lambda_2)}{d\lambda}\right|_{\lambda_2=-1} = \left.\frac{d}{d\lambda_2}(e^{\lambda_2 t})\right|_{\lambda_2=-1} = te^{-t},$$

the above equations in the matrix form are

$$\begin{bmatrix} 1 & -7/2 & 49/4 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} e^{-3.5t} \\ e^{-t} \\ te^{-t} \end{bmatrix}.$$

The solution for β's gives

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0.16e^{-3.5t} + 0.84e^{-t} + 1.4te^{-t} \\ 0.32e^{-3.5t} - 0.32e^{-t} + 1.8te^{-t} \\ 0.16e^{-3.5t} - 0.16e^{-t} + 0.4te^{-t} \end{bmatrix}.$$

With β's known, the desired matrix is

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}\beta_0 + \begin{bmatrix} -3.5 & 0.5 & -0.5 \\ 0.25 & -1.5 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}\beta_1$$

$$+ \begin{bmatrix} 12.125 & -2.75 & 1.875 \\ -1.375 & 2.25 & 0.375 \\ -1.875 & -0.75 & -0.125 \end{bmatrix}\beta_2.$$

Substituting the β's from the previous solution, and after simplifying, we obtain

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 0.98e^{-3.5t}+0.02e^{-t}-0.05te^{-t} & -0.28e^{-3.5t}+0.28e^{-t}-0.2te^{-t} & 0.14e^{-3.5t}-0.14e^{-t}-0.15te^{-t} \\ -0.14e^{-3.5t}+0.14e^{-t}-0.1te^{-t} & 0.04e^{-3.5t}+0.96e^{-t}-0.4te^{-t} & -0.02e^{-3.5t}+0.02e^{-t}-0.3te^{-t} \\ -0.14e^{-3.5t}+0.14e^{-t}+0.15te^{-t} & 0.04e^{-3.5t}-0.04e^{-t}+0.6te^{-t} & 1.02e^{-3.5t}-0.02e^{-t}+0.45te^{-t} \end{bmatrix}.$$

(c) *Complex eigenvalues*

We shall illustrate the computation of a matrix exponential when some of the roots of the characteristic equation are complex quantities, considering the following example.

Example 5.8

Let the circuit in Fig. 5.11 (of the previous example) have the same parameters, excluding $R_6 = 2/5\ \Omega$ and $R_7 = 1/2\ \Omega$. Our purpose is again to compute $\mathbf{e}^{\mathbf{A}t}$.

Solution

We substitute the above parameters into the **A** matrix of equation 5.55 to yield

$$\mathbf{A} = \begin{bmatrix} -3 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus, the characteristic equation of **A** is

$$g(\lambda) = (\lambda+3)(\lambda+\tfrac{5}{4})(\lambda+\tfrac{1}{2}) + \tfrac{1}{4}\lambda + \tfrac{3}{4} = 0,$$

or after a rearrangement of terms

$$(\lambda+3)(\lambda^2+\tfrac{7}{4}\lambda+\tfrac{7}{8}) = 0,$$

Therefore, the eigenvalues are

$$\lambda_1 = -3, \quad \lambda_{2,3} = -\tfrac{7}{8} \pm \sqrt{\tfrac{49-56}{64}} = -0.875 \pm j0.331.$$

Note that two complex eigenvalues are a conjugate pair. Thus, in accordance with equation 5.106, we have

$$\beta_0 + \beta_1(-3) + \beta_2(-3)^2 = e^{-3t}$$
$$\beta_0 + \beta_1(-0.875 + j0.331) + \beta_2(-0.875 + j0.331)^2 = e^{-0.875t}\, e^{j0.331t}$$
$$\beta_0 + \beta_1(-0.875 - j0.331) + \beta_2(-0.875 - j0.331)^2 = e^{-0.875t}\, e^{-j0.331t}.$$

Next, we solve these equations to yield for β's:

$$\beta_0 = 0.819e^{-3t} + e^{-0.875t}(3.86 \sin 0.331t + 0.811 \cos 0.331t)$$
$$\beta_1 = 0.378e^{-3t} + e^{-0.875t}(5.46 \sin 0.331t - 0.378 \cos 0.331t)$$
$$\beta_2 = 0.216e^{-3t} + e^{-0.875t}(1.39 \sin 0.331t - 0.216 \cos 0.331t).$$

Hence, matrix $\mathbf{e}^{\mathbf{A}t}$ will be

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}\beta_0 + \begin{bmatrix} -3 & 0.5 & -0.5 \\ 0.25 & -1.25 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}\beta_1$$
$$+ \begin{bmatrix} 8.875 & -2.375 & 1.625 \\ -1.187 & 1.563 & 0.313 \\ -1.625 & -0.625 & -1.125 \end{bmatrix}\beta_2.$$

Finally, substituting the above results for β's, after simplifying, we obtain

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 0.973e^{-3t} - 0.174\zeta_1 + 0.027\zeta_2 & -0324e^{-3t} - 0.572\zeta_1 + 0.324\zeta_2 & 0.162e^{-3t} - 0.470\zeta_1 - 0.162\zeta_2 \\ -0.162e^{-3t} - 0.280\zeta_1 + 0.162\zeta_2 & 0.054e^{-3t} - 0.787\zeta_1 + 0.946\zeta_2 & -0.027e^{-3t} - 0.930\zeta_1 + 0.027\zeta_2 \\ -0.162e^{-3t} + 0.470\zeta_1 + 0.162\zeta_2 & -0.054e^{-3t} + 1.86\zeta_1 - 0.054\zeta_2 & -0.027e^{-3t} + 0.960\zeta_1 + 1.027\zeta_2 \end{bmatrix}$$

where $\zeta_1 = e^{-0.875t} \sin 0.331t$, $\zeta_2 = e^{-0.875t} \cos 0.331t$.

Suppose we now wish to know the zero input response of the circuit to the initial vector, $\mathbf{x}(0) = [1 \;\; 1 \;\; 0]^T$, i.e. the capacitors are initially charged to 1 V each. Then,

$$\mathbf{x}_{nat}(t) = \mathbf{e}^{\mathbf{A}t}[1 \quad 1 \quad 0]^T = \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}$$
$$= \begin{bmatrix} 0.649e^{-3t} + e^{-0.875t}(-0.746 \sin 0.331t + 0.351 \cos 0.331t) \\ -0.108e^{-3t} + e^{-0.875t}(-1.073 \sin 0.331t + 1.108 \cos 0.331t) \\ -0.108e^{-3t} + e^{-0.875t}(2.329 \sin 0.331t + 0.108 \cos 0.331t) \end{bmatrix}.$$

These two voltage curves and one current curve versus time are shown in Fig. 5.13.

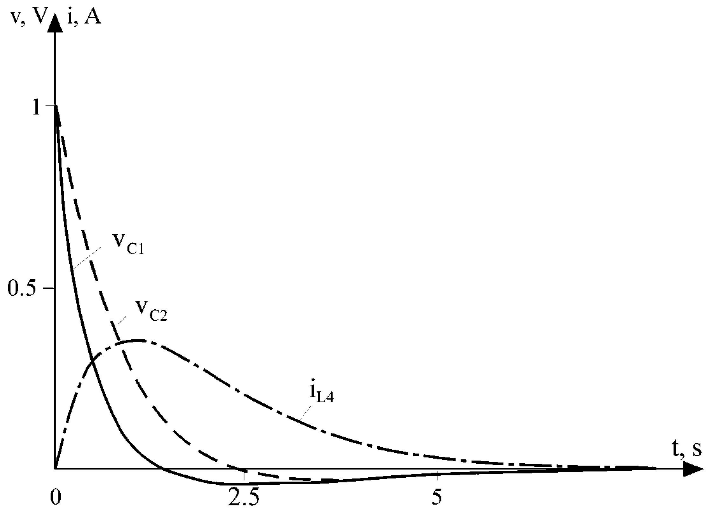

Figure 5.13 Two capacitor voltages and inductor current curves versus time of Example 5.8 in the case of complex-conjugate eigenvalues.

5.8.3 Lagrange interpolation formula

One other method of computing functions of a matrix is based on the Lagrange interpolation formula (this formula is also known as the Silvestre formula). Thus, knowing the eigenvalues λ's of matrix $\mathbf{A}$, any function of $\mathbf{A}$ may be determined as:

$$f(\mathbf{A}) = \sum_{i=1}^{n} \left(\prod_{\substack{k=1 \\ k\neq 1}}^{n} \frac{\mathbf{A} - \lambda_k \mathbf{1}}{\lambda_i - \lambda_k} \right) f(\lambda_i), \tag{5.127}$$

where $\prod_{\substack{k=1 \\ k\neq 1}}^{n}$ means the product of terms $\dfrac{\mathbf{A} - \lambda_k \mathbf{1}}{\lambda_i - \lambda_k}$ where k takes the values $1, 2, \ldots, n$ but excluding $k = i$. For example, using the data of Example 5.6, equation 5.127 implies that

$$\mathbf{e}^{\mathbf{A}t} = \frac{(\mathbf{A} + 1.5\cdot\mathbf{1})(\mathbf{A} + 4\cdot\mathbf{1})}{(-0.75 + 1.5)(-0.75 + 4)} e^{-0.75t} + \frac{(\mathbf{A} + 0.75\cdot\mathbf{1})(\mathbf{A} + 4\cdot\mathbf{1})}{(-1.5 + 0.75)(-1.5 + 4)} e^{-1.5t}$$
$$+ \frac{(\mathbf{A} + 0.75\cdot\mathbf{1})(\mathbf{A} + 1.5\cdot\mathbf{1})}{(-4 + 0.75)(-4 + 1.5)} e^{-4t}.$$

Substituting matrix $\mathbf{A}$ (equation 5.110) and performing all the arithmetic, leads to

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} -0.050 & -0.154 & -0.256 \\ -0.077 & -0.230 & -0.385 \\ 0.256 & 0.769 & 1.282 \end{bmatrix} e^{-0.75t} + \begin{bmatrix} 0.067 & 0.4 & 0.133 \\ 0.2 & 1.2 & 0.4 \\ -0.133 & -0.8 & -0.267 \end{bmatrix} e^{-1.5t}$$

$$+ \begin{bmatrix} 0.985 & -0.246 & 0.123 \\ -0.123 & 0.031 & -0.015 \\ -0.123 & 0.031 & -0.015 \end{bmatrix} e^{-4t}$$

which agrees with the previous results obtained in equation 5.115.

The Lagrange interpolation formula can be easily programmed, which is an advantage in computer-aided calculations.

5.9 EVALUATING THE MATRIX EXPONENTIAL BY LAPLACE TRANSFORM

In conclusion, let us introduce the Laplace transform application for solving the matrix differential equation. To simplify the procedure, we first apply the Laplace transform to the homogeneous equation (see equation 5.81):

$$\frac{d}{dt}\mathbf{x}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}. \tag{5.128}$$

Applying the Laplace transform to equation 5.128, we get

$$s\mathbf{X}(s) - \mathbf{X}(0) - \mathbf{A}\mathbf{X}(s) = \mathbf{0}, \tag{5.129}$$

where $\mathbf{X}(s)$ is the Laplace transform of $\mathbf{x}(t)$. Supposing that $\mathbf{X}(0) = 1$ (equation 5.129) can be written as follows:

$$(s \cdot \mathbf{1} - \mathbf{A})\mathbf{X}(s) = \mathbf{1}, \tag{5.130}$$

or

$$\mathbf{X}(s) = (s \cdot \mathbf{1} - \mathbf{A})^{-1}. \tag{5.131}$$

Now, we take the inverse transform to get $\mathbf{x}(t)$

$$\mathbf{x}(t) = L^{-1}\{(s \cdot \mathbf{1} - \mathbf{A})^{-1}\} = \mathbf{e}^{\mathbf{A}t}. \tag{5.132}$$

As can be seen, since we have taken $\mathbf{X}(0) = \mathbf{1}$, this expression is also equal to the matrix exponential $\mathbf{e}^{\mathbf{A}t}$.

Example 5.9

Let us apply this result to the simple circuit shown in Fig. 5.14, where the proper tree branches are emphasized.

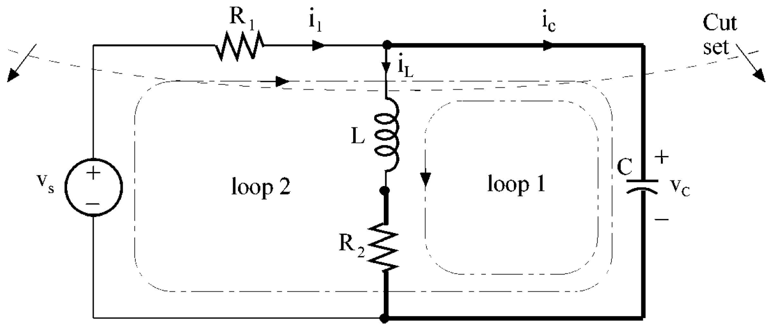

Figure 5.14 A circuit of Example 5.9.

Solution

The capacitor voltage v_C and the inductor current i_L are the state variables in this case. The fundamental cut-set equation and two fundamental loop equations yield

$$C\frac{dv_C}{dt} = -i_L + i_1$$

$$L\frac{di_L}{dt} = v_C - R_2 i_L$$

$$R_1 i_1 = -v_C + v_s \quad \text{or} \quad i_1 = -\frac{1}{R_1}v_C + \frac{1}{R_1}v_s.$$

To eliminate a non-desirable variable, i_1, in the first equation, in this simple case, the third equation shall be inserted into the first one for i_1. Thus, the state equations are

$$\frac{dv_C}{dt} = -\frac{1}{R_1 C}v_C - i_L + \frac{1}{R_1}v_s$$

$$\frac{di_L}{dt} = \frac{1}{L}v_C - \frac{R_2}{L}i_L,$$

or in the matrix form

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -1/R_1 C & -1 \\ 1/L & -R_2/L \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/R_1 \\ 0 \end{bmatrix}[v_s]. \tag{5.133}$$

Let the element values be $C = 1.0$ F, $L = 4/3$ H, $R_1 = 2/5\ \Omega$, $R_2 = 2/3\ \Omega$ and $v_s =$ 1 V. This yields the coefficient matrixes **A** and **b**

$$\mathbf{A} = \begin{bmatrix} -5/2 & -1 \\ 3/4 & -1/2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5/2 \\ 0 \end{bmatrix} \tag{5.134}$$

and the input matrix $\mathbf{w} = [v_s] = [1]$. Next, we find the matrix $[s\mathbf{1} - \mathbf{A}]$ and its determinant

$$s\mathbf{1} - \mathbf{A} = \begin{bmatrix} s + \frac{5}{2} & 1 \\ -\frac{3}{4} & s + \frac{1}{2} \end{bmatrix}$$

$$\det(s\mathbf{1} - \mathbf{A}) = (s + \tfrac{5}{2})(s + \tfrac{1}{2}) + \tfrac{3}{4} = s^2 + 3s + 2 = (s + 1)(s + 2).$$

The inverse matrix $[s \cdot \mathbf{1} - \mathbf{A}]^{-1}$ is now easily obtained as

$$[s \cdot \mathbf{1} - \mathbf{A}] = \begin{bmatrix} \dfrac{s + \frac{1}{2}}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ -\dfrac{\frac{3}{4}}{(s+1)(s+2)} & \dfrac{s + \frac{5}{2}}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{\frac{1}{2}}{s+1} + \dfrac{\frac{3}{2}}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ -\dfrac{\frac{3}{4}}{s+1} + \dfrac{\frac{3}{4}}{s+2} & \dfrac{\frac{3}{2}}{s+1} - \dfrac{\frac{1}{2}}{s+2} \end{bmatrix}.$$

A partial-fraction expansion was performed in the last step. The inverse Laplace transform of this expression is

$$L^{-1}[s \cdot \mathbf{1} - \mathbf{A}]^{-1} = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} & e^{-t} - e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-t} \end{bmatrix} = \mathbf{e}^{\mathbf{A}t}. \tag{5.135}$$

(It is left as an exercise for the reader to verify this result using one of the above given methods for determining a matrix exponential.)

Suppose that the initial conditions are $v_C = 1$ V and $i_L(0) = 0$, and then the natural response will be

$$\mathbf{x}_n(t) = \begin{bmatrix} v_{C,n} \\ i_{L,n} \end{bmatrix} = \mathbf{e}^{\mathbf{A}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} \end{bmatrix}. \tag{5.136}$$

Note that the verification of equation 5.136 at $t = 0$ yields the initial values of $v_C(0)$ and $i_L(0)$. The particular solution of equation 5.133 may also be obtained with equation 5.135 using, for example, equation 5.118. Thus,

$$\mathbf{x}_p(t) = \mathbf{A}^{-1}[\mathbf{e}^{\mathbf{A}t} - \mathbf{1}]\mathbf{bw} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ -\frac{3}{8} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} & e^{-t} - e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-2t} \end{bmatrix} \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}$$

or after performing all the calculations

$$\mathbf{x}_{part}(t) \begin{bmatrix} v_{C,p} \\ i_{L,p} \end{bmatrix} = \begin{bmatrix} \frac{5}{4}e^{-t} - \frac{15}{8}e^{-2t} + \frac{5}{8} \\ -\frac{15}{8}e^{-t} + \frac{15}{16}e^{-2t} + \frac{15}{16} \end{bmatrix}.$$

By inspection (see the circuit in Fig. 5.13) it can be easily verified that the steady-state values of the capacitor voltage and the inductor current agree with those found below:

$$v_{C,p(\infty)} = \tfrac{5}{8}\,\mathrm{V} \quad \text{and} \quad i_{L,p(\infty)} = \tfrac{15}{16}\,\mathrm{A}.$$

The Laplace transform is one of the ways of evaluating the matrix exponential. However, if we are going to use the Laplace transform for circuit analysis, we may do it straightforwardly using the methods described in Chapter 3. The methods of matrix function evaluation, considered in this chapter, are the most general and suitable for computer-aided computation.