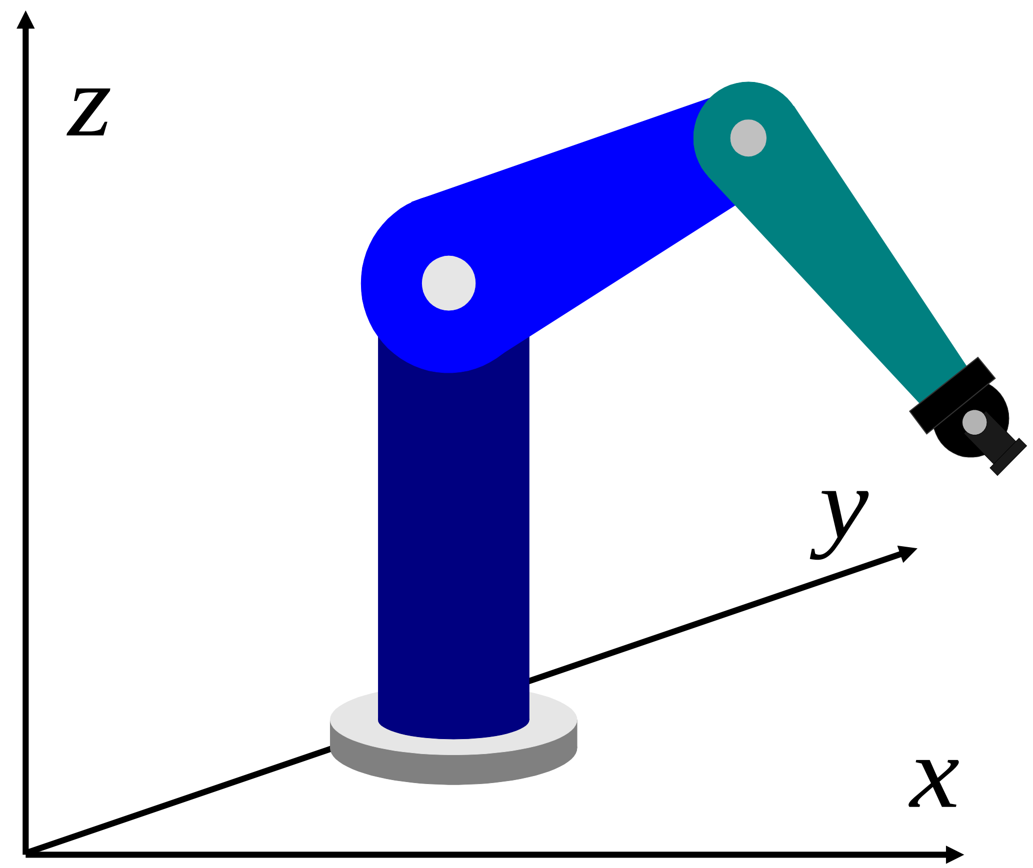
Inverse Kinematics

Given a desired position (**p**) & orientation (**R**) of the end-effector

$$q = (q_1, q_2, \cdots q_n)$$

Find the joint variables which can bring the robot the desired configuration



The Inverse Kinematics Problem

- Direct Kinematics

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q})$$

- Inverse Kinematics

$$\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{x})$$

The Inverse Kinematics Problem

- The problem is not simple!
- **A general approach for the solution of this problem <u>does not exist</u>**
- On the other hand, for the most common kinematic structures, a scheme for obtaining the solution has been found. Unfortunately…
- …**The solution is <u>not unique</u>**
- In general we may have:
  - ➢ No solution (e.g. starting with a position x not in the workspace)
  - ➢ A finite set of solutions (one or more)
  - ➢ Infinite solutions
- We seek for closed form solutions not based on numerical techniques:
  - The analytic solution is more efficient from the computational point of view
  - If the solutions are known analytically, it is possible to select one of them on the basis of proper criteria.

The Inverse Kinematics Problem
Difficulties

Possible Problems of Inverse Kinematics

- Nonlinear (Revolute joints → inverse trigonometry)
- Discontinuities and singularities
- Can lose one or more DOFs in some configurations
- Multiple solutions for a single Cartesian pose
- Infinitely many solutions
- Possibly no solutions
- No closed-form (analytical) solutions
- Not enough!! [Dynamics: in reality, we want to apply forces and torques (while respecting physical constraints), not just move arm!]

The Inverse Kinematics Problem
What have we swept under the rug?

- Sensing
  - Shape, pose of target object, accessibility of surfaces
  - Classification of material type from sensor data
  - Freespace through which grasping action will occur
- Prior knowledge
  - Estimate of mass, moments given material type
  - Internal, articulated, even active degrees of freedom
- Uncertainty & compliance
  - Tolerate noise inherent in sensing and actuation
  - Ensure that slight sensing, actuation errors won’t cause damage
  - Handle soft fingers making contact over a finite area (not a point)
- Dynamics
  - All of the above factors may be changing in real time

Algebraic Approach

For a 6 DOF manipulator, the kinematic model is described by the equation

$$T_0^{n=6} = T_0^1(q_1)T_1^2(q_2)\cdots T_{n-1=5}^{n=6}(q_{n=6})$$

equivalent to 12 equations in the 6 unknowns q_i , $i = 1, \ldots, 6$.

Example: spherical manipulator (only 3 DOF)

$$\mathbf{T} = \begin{bmatrix} 0.5868 & -0.6428 & 0.4394 & -0.4231 \\ 0.5265 & 0.7660 & 0.3687 & 0.9504 \\ -0.5736 & 0.0000 & 0.8192 & 0.4096 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 & -d_2 S_1 + d_3 C_1 S_2 \\ C_2 S_1 & C_1 & S_1 S_2 & d_2 C_1 + d_3 S_1 S_2 \\ -S_2 & 0 & C_2 & d_3 C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since both the numerical values and the structure of the **intermediate matrices** are known, then by suitable pre- / post-multiplications it is possible to obtain equations

$$\left[T_0^1(q_1)T_1^2(q_2)\cdots T_{i-1}^i(q_i)\right]^{-1} T_0^n = T_i^{i+1}(q_{i+1})T_{i+1}^{i+2}(q_{i+2})\cdots T_{n-1}^n(q_n)$$

There will be 12 new equations for each i, covering the range 1 to n.

Then, by selecting the most simple equations among all those obtained, **it might be possible** to obtain a solution to the problem.

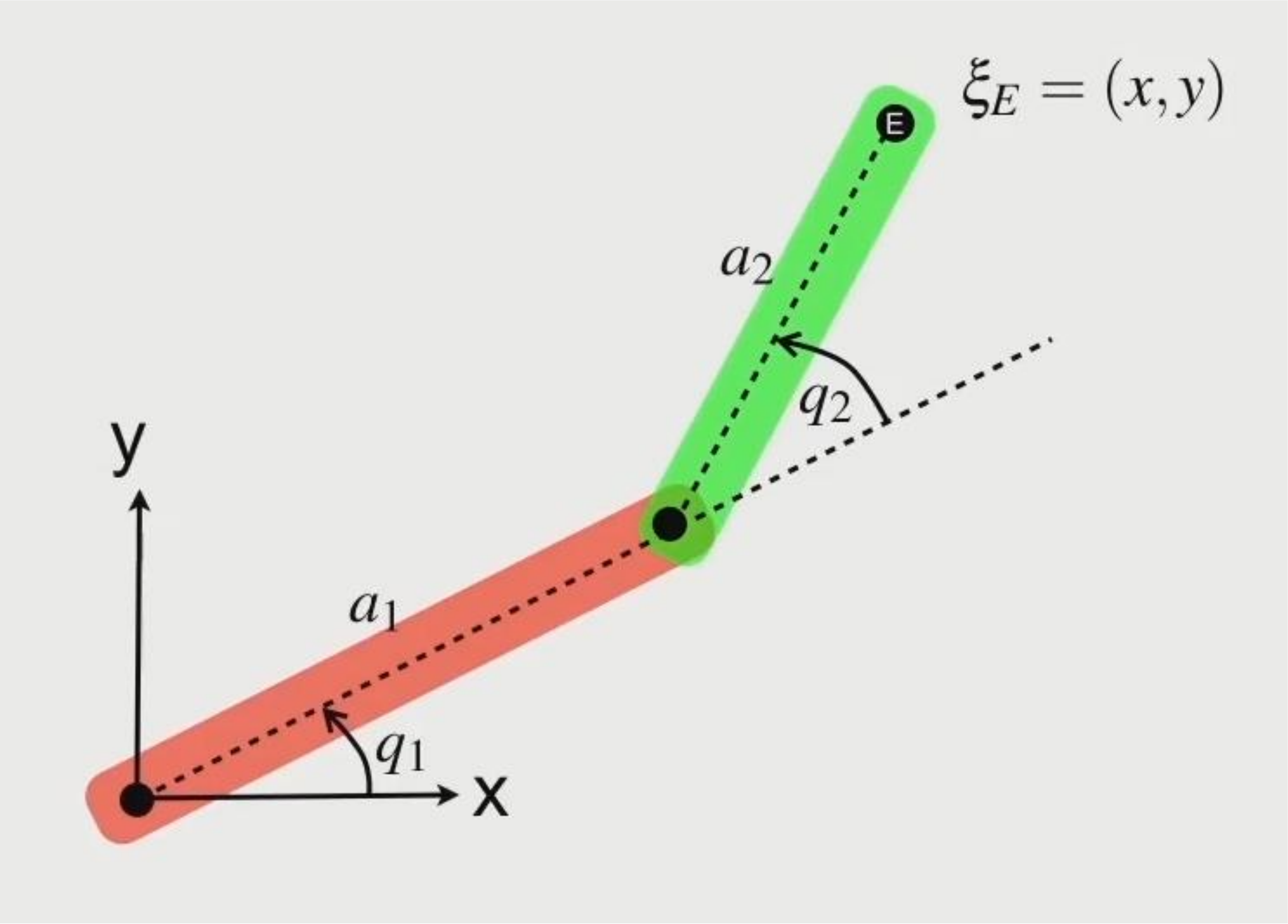
The other way: use geometry

It may be possible to exploit considerations related to the **geometrical structure** of the manipulator

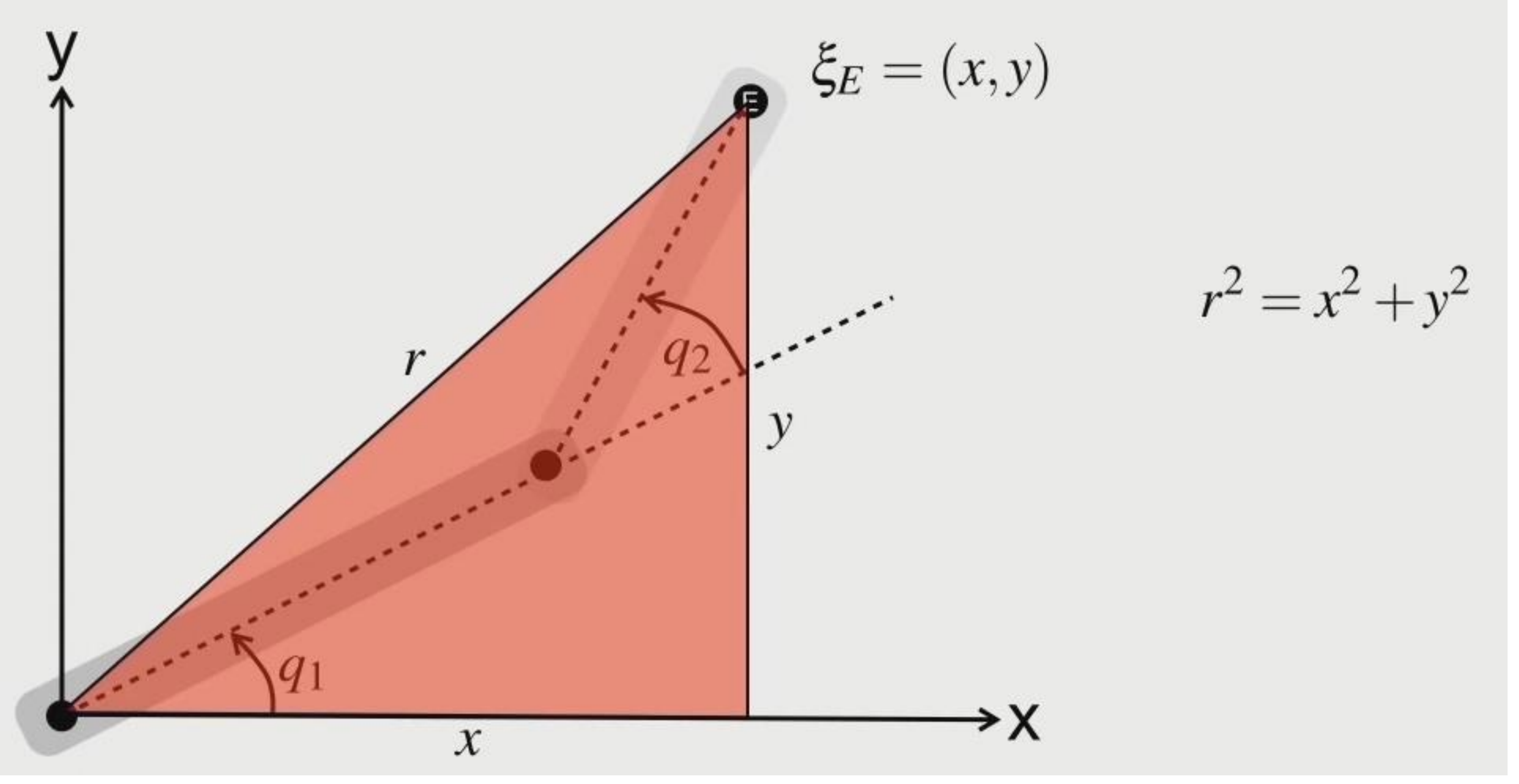
Example 1: the 2 DOF arm

Analytical Inverse Kinematics of a 2 DOF Arm

Geometric approach

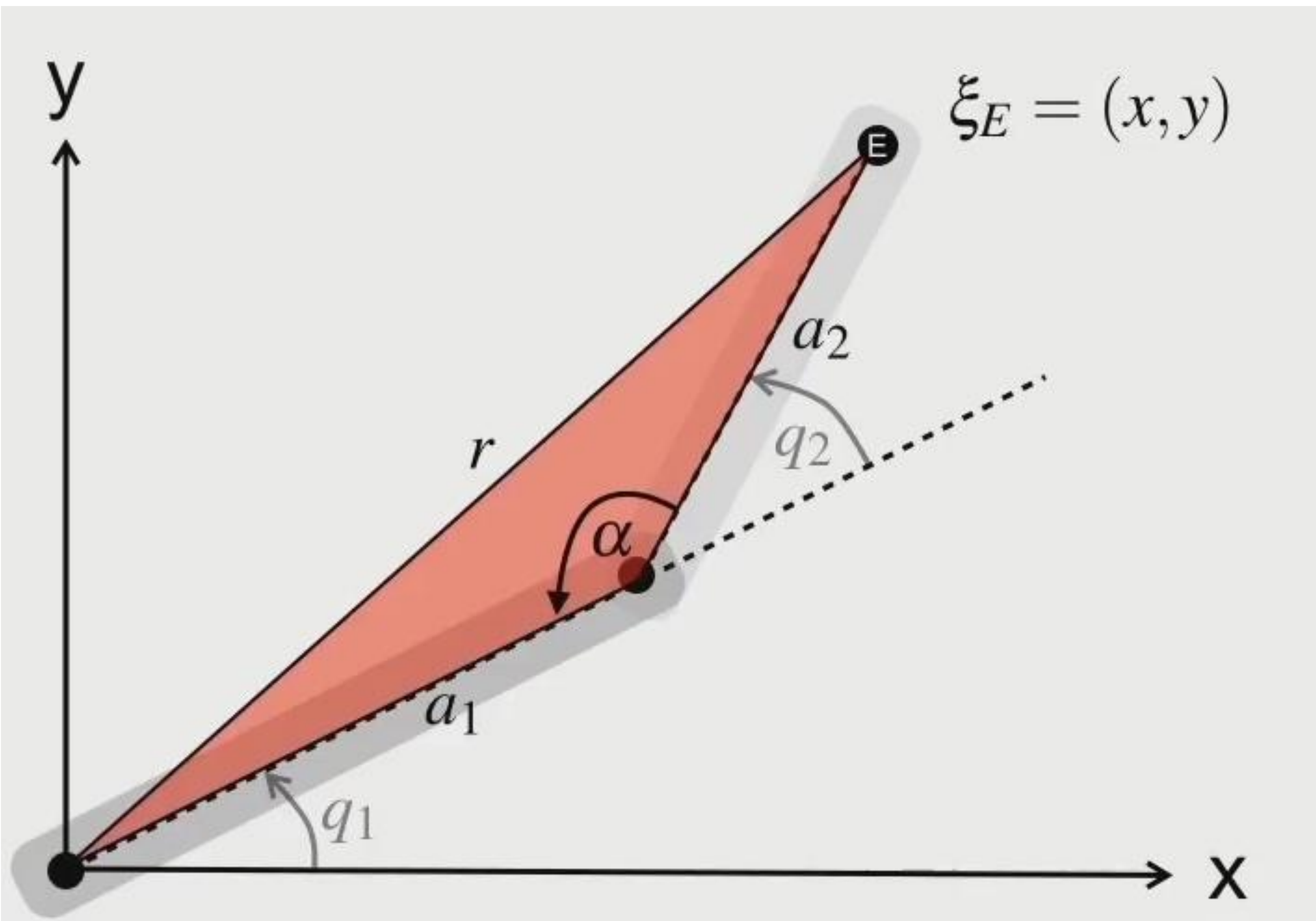


Analytical Inverse Kinematics of a 2 DOF Arm



$$r^2 = x^2 + y^2$$

Analytical Inverse Kinematics of a 2 DOF Arm



$$r^2 = x^2 + y^2$$

$$r^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos\alpha$$

$$\cos\alpha = \frac{a_1^2 + a_2^2 - r^2}{2a_1 a_2}$$

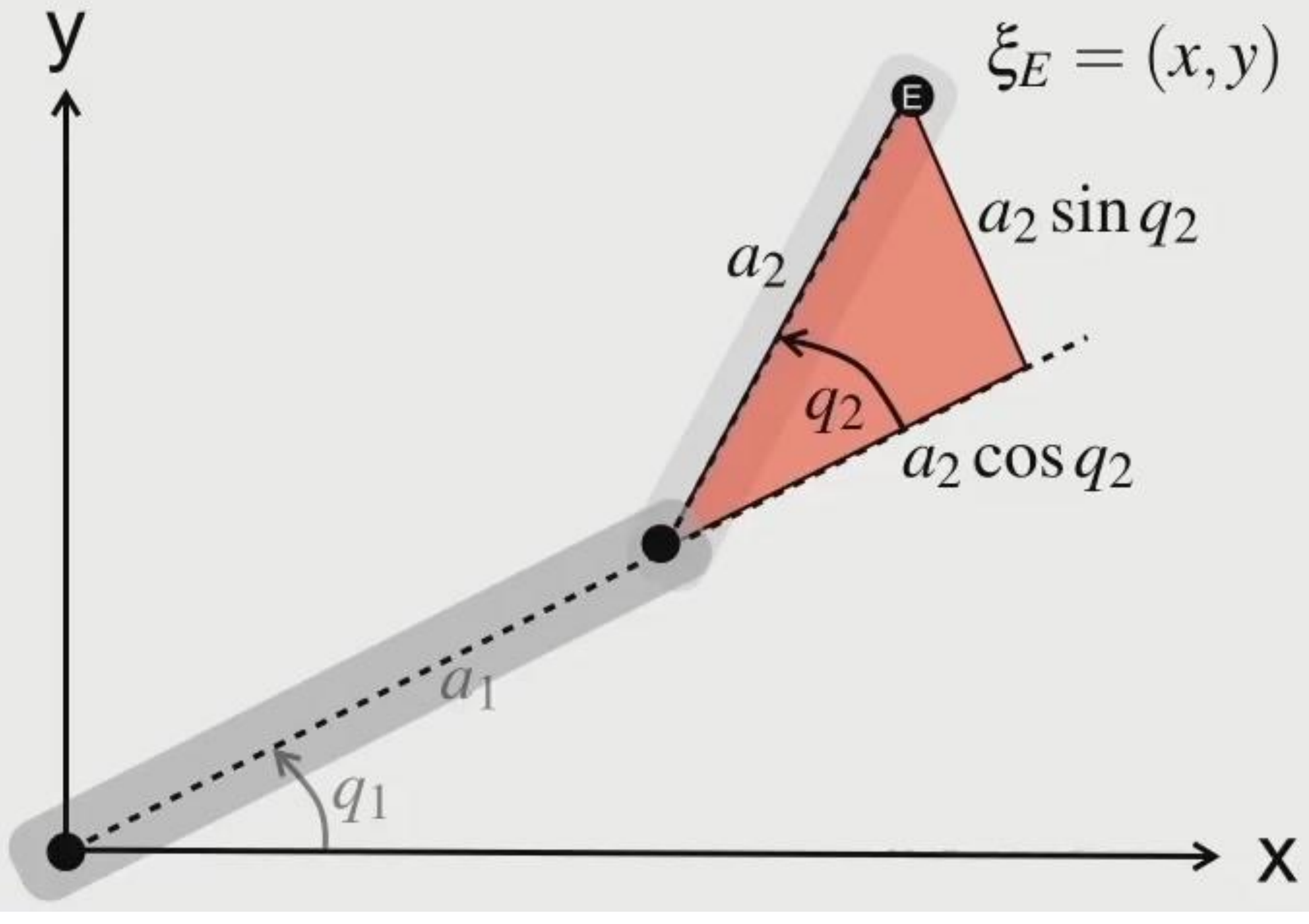
$$= \frac{a_1^2 + a_2^2 - x^2 - y^2}{2a_1 a_2}$$

$$q_2 = \pi - \alpha$$

$$\cos q_2 = -\cos\alpha$$

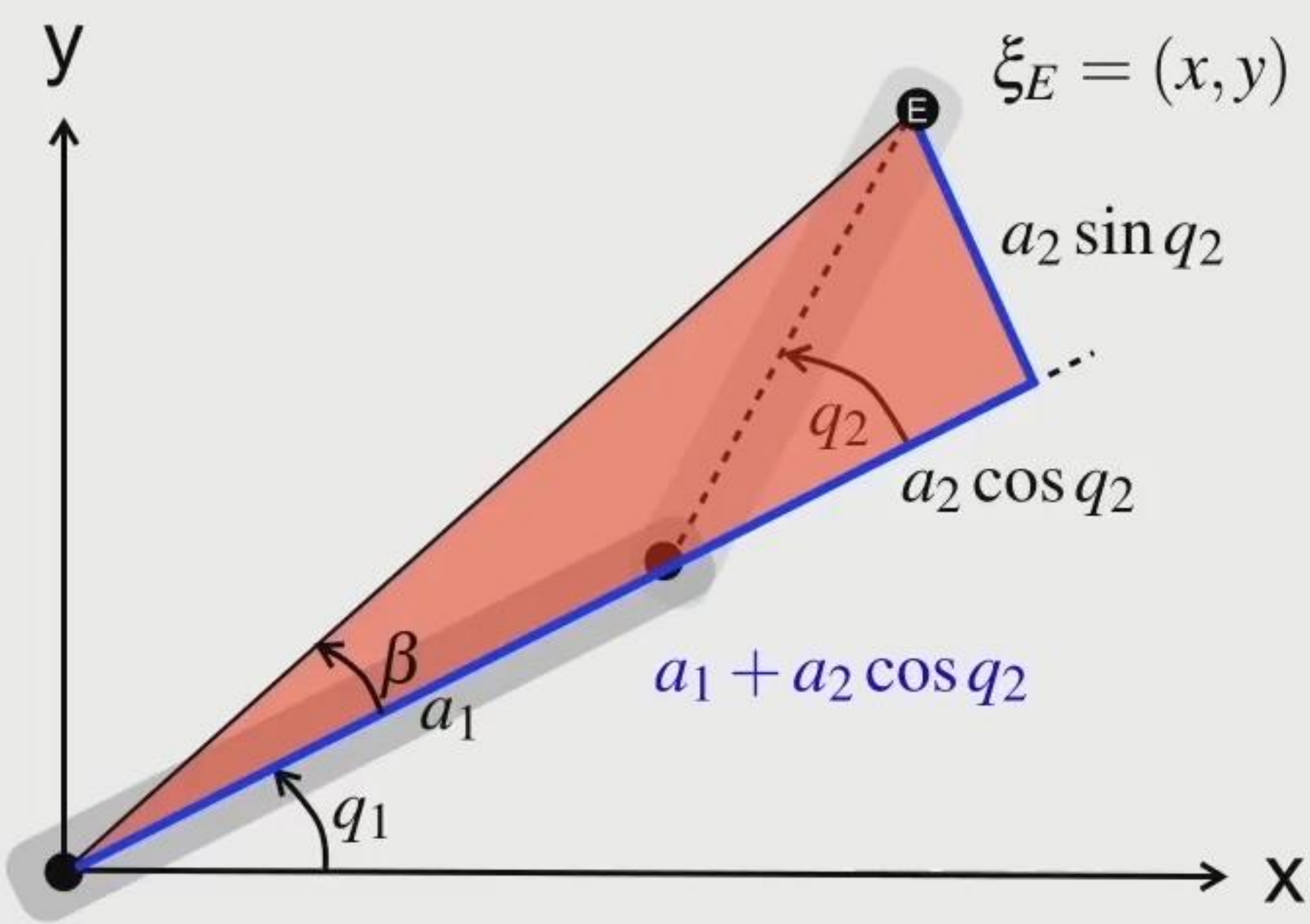
$$\cos q_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm



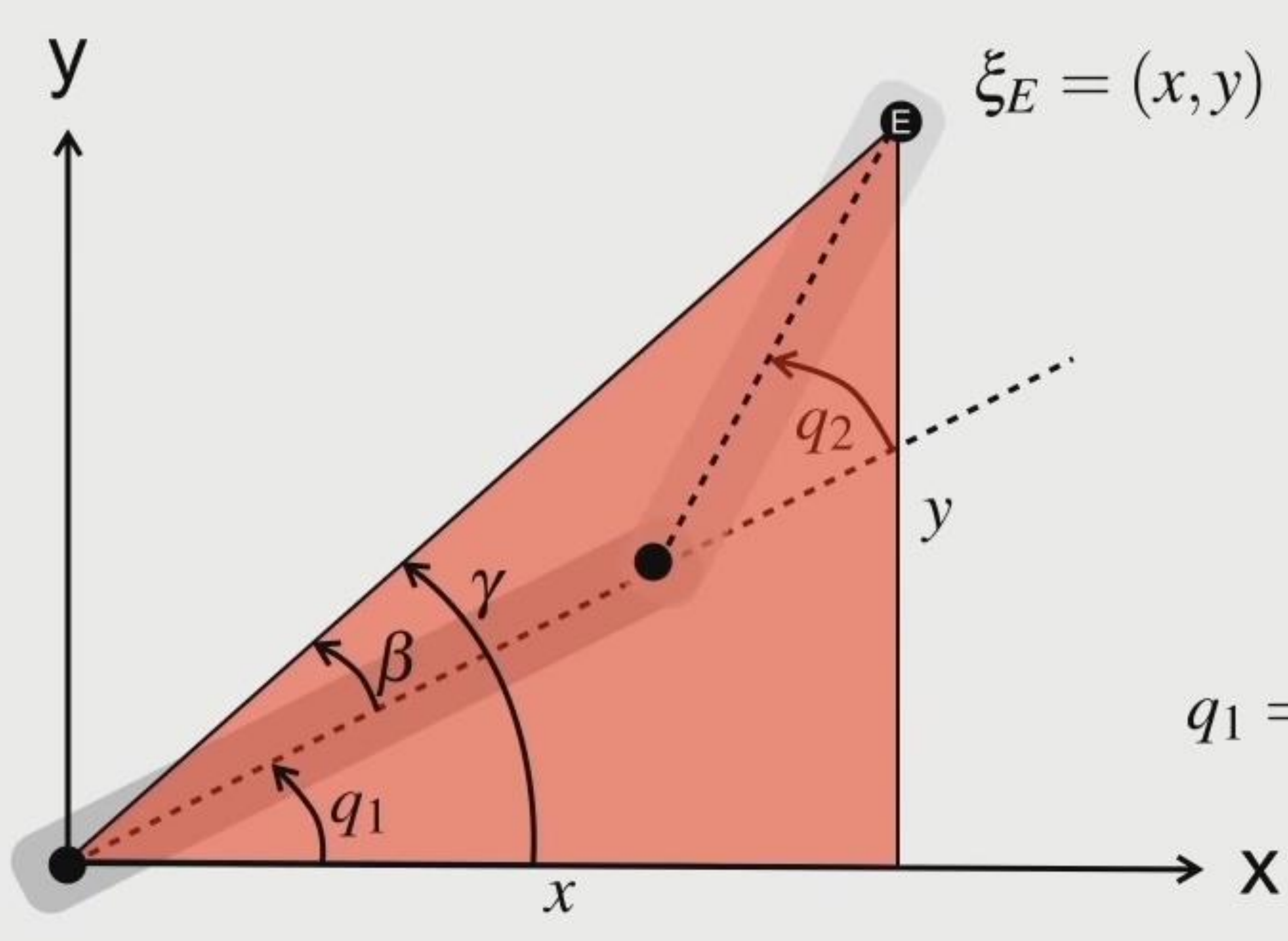
$$\sin q_2 = \sqrt{1 - \cos q_2^2}$$

Analytical Inverse Kinematics of a 2 DOF Arm



$$\beta = \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm



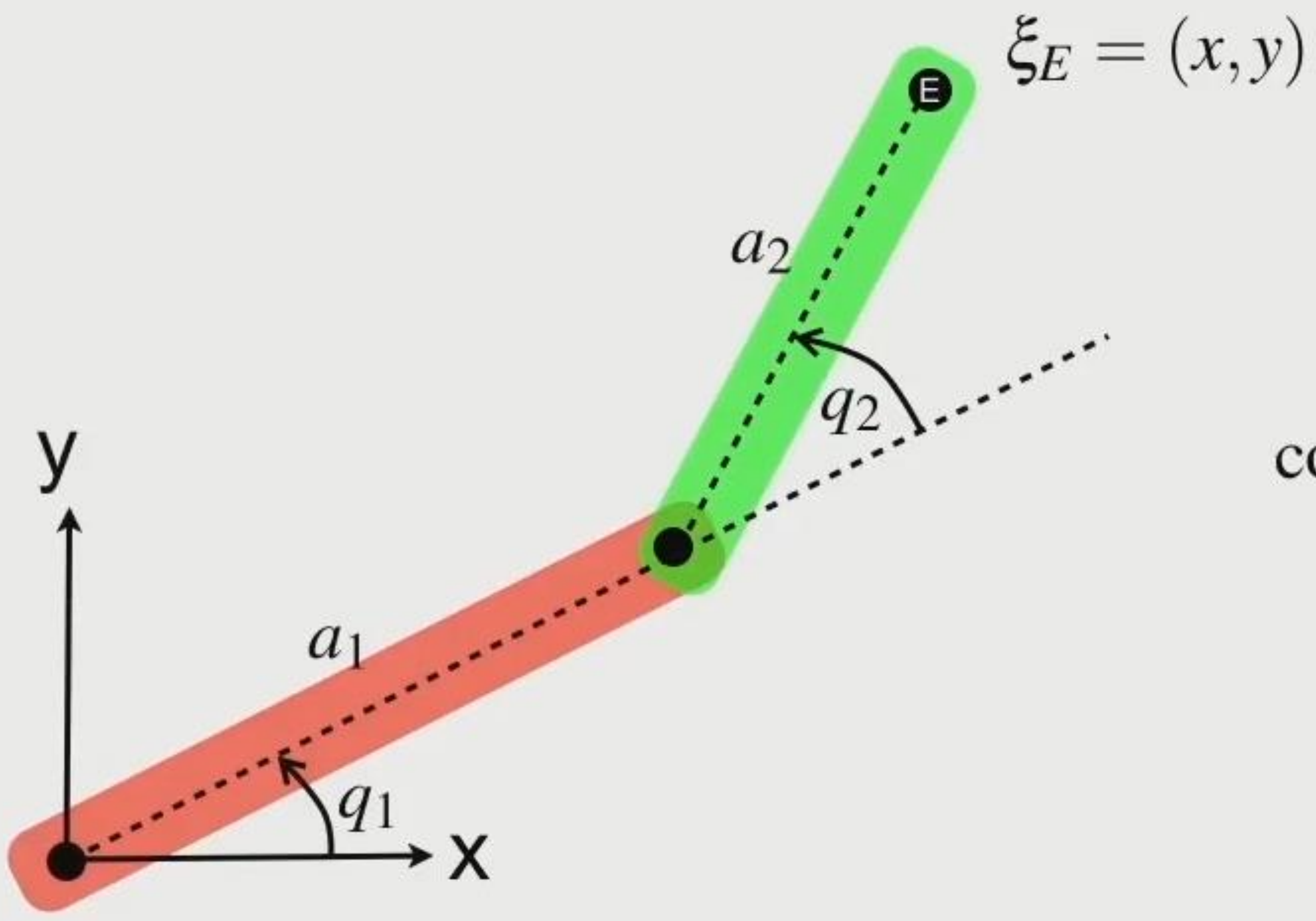
$$\gamma = \tan^{-1} \frac{y}{x}$$

$$q_1 = \gamma - \beta$$

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm

Partial results

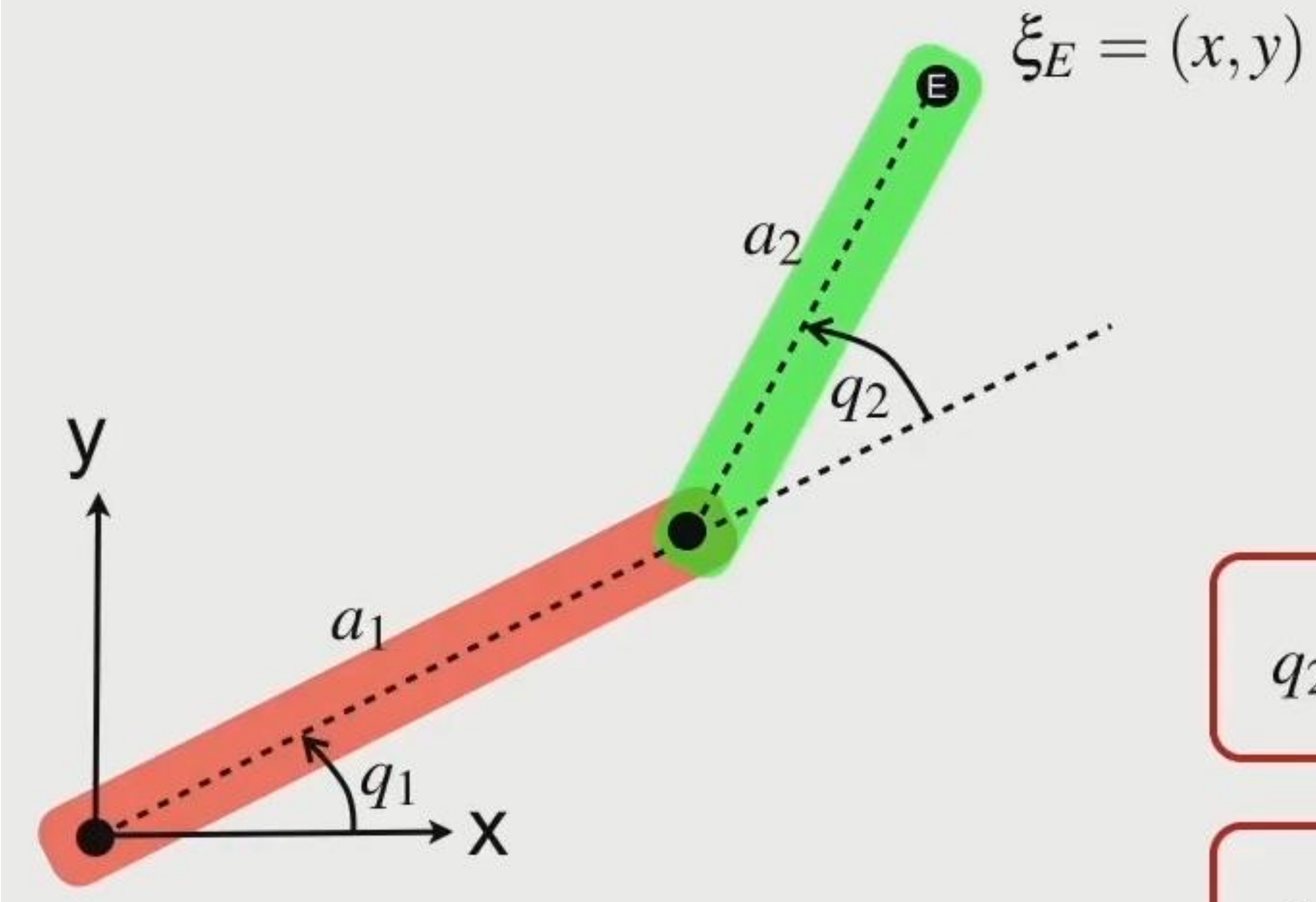


$$\cos q_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$q_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm

Solution for positive angle q_2



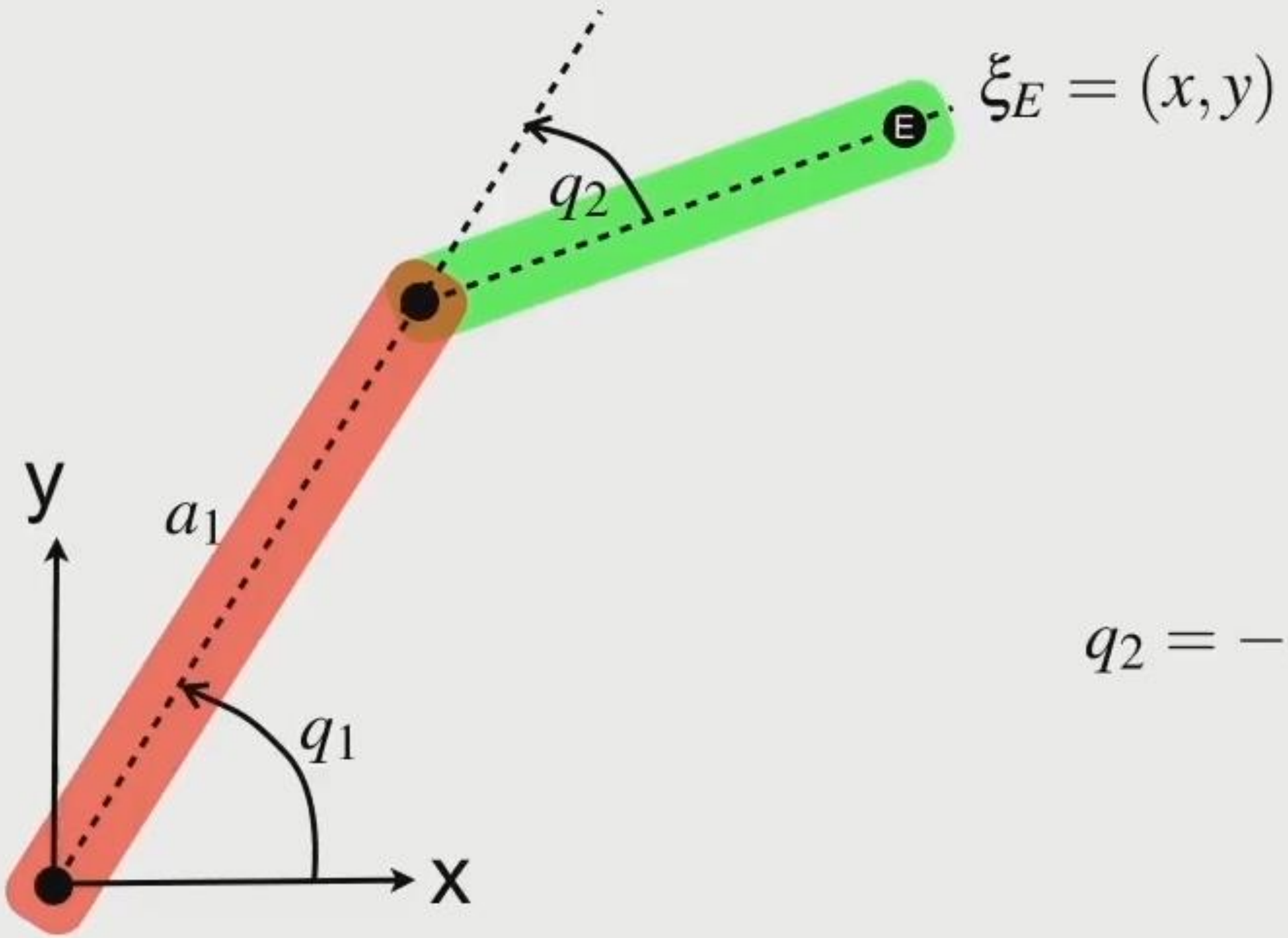
$$q_2 = \cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2 a_1 a_2}$$

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Not independent!

Analytical Inverse Kinematics of a 2 DOF Arm

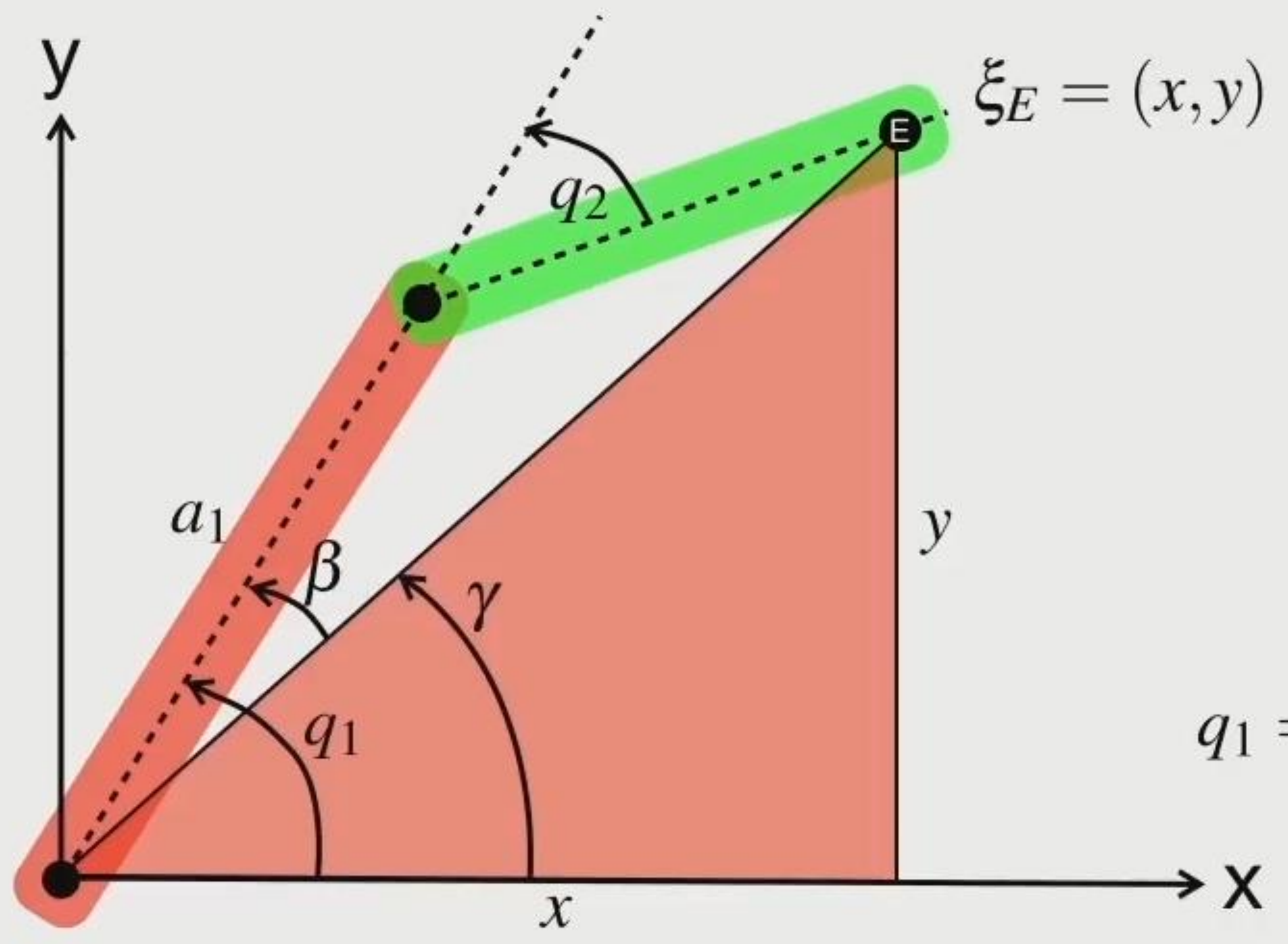
Case of negative angle q_2



$$q_2 = -\cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm

Solve for q_1



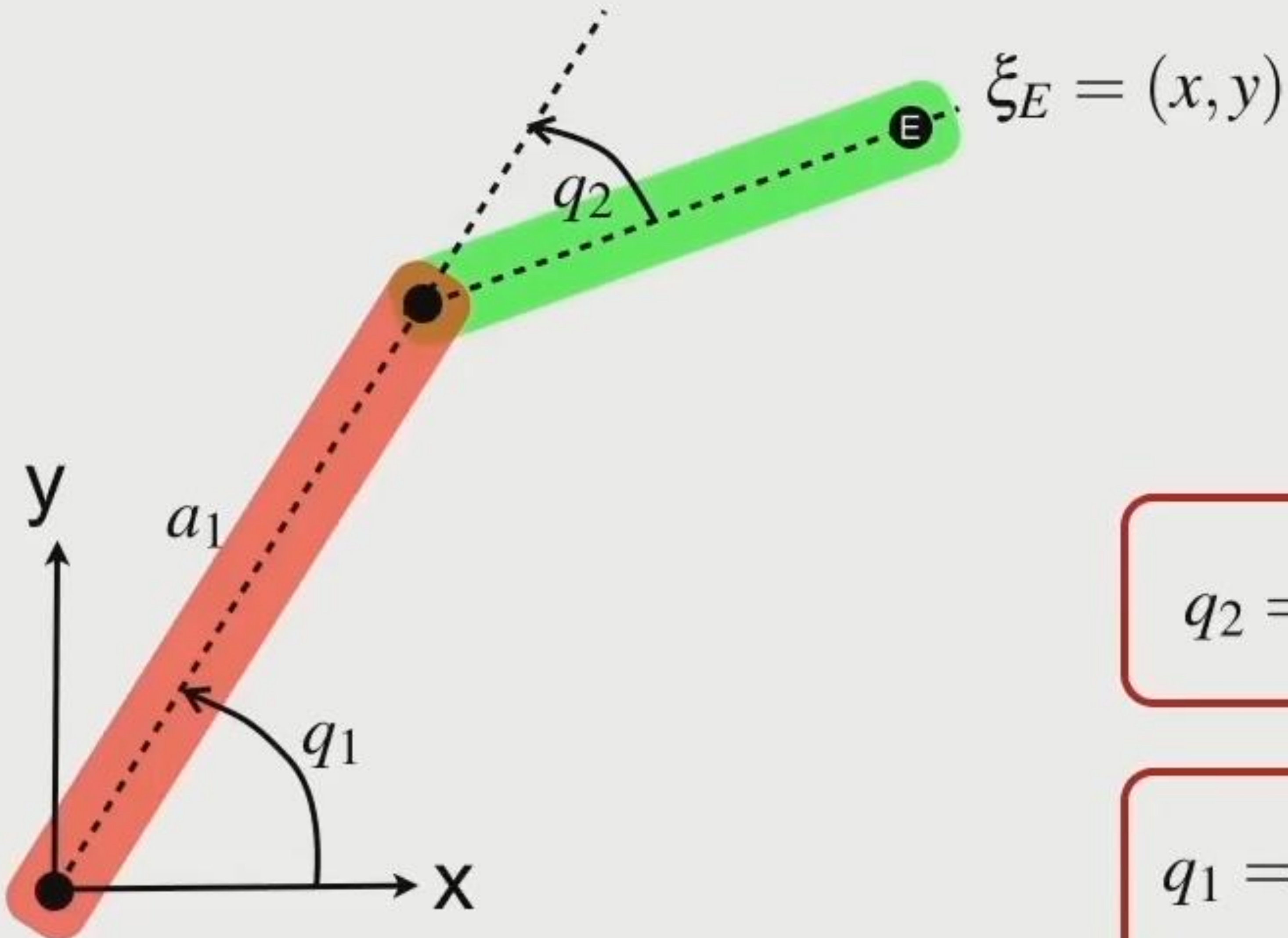
$$\gamma = \tan^{-1}\frac{y}{x}$$

$$q_1 = \gamma + \beta$$

$$q_1 = \tan^{-1}\frac{y}{x} + \tan^{-1}\frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm

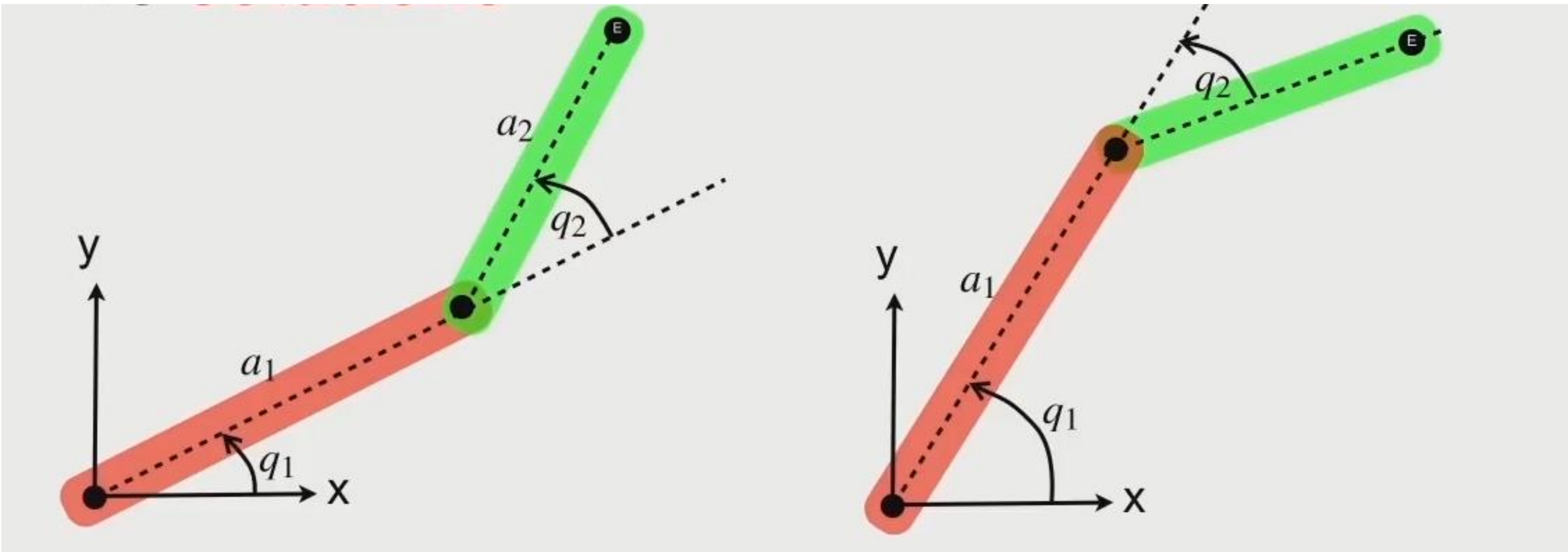
Solution for negative angle q_2



$$q_2 = -\cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2 a_1 a_2}$$

$$q_1 = \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

Analytical Inverse Kinematics of a 2 DOF Arm



$$q_2 = \cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$q_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

$$q_2 = -\cos^{-1} \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

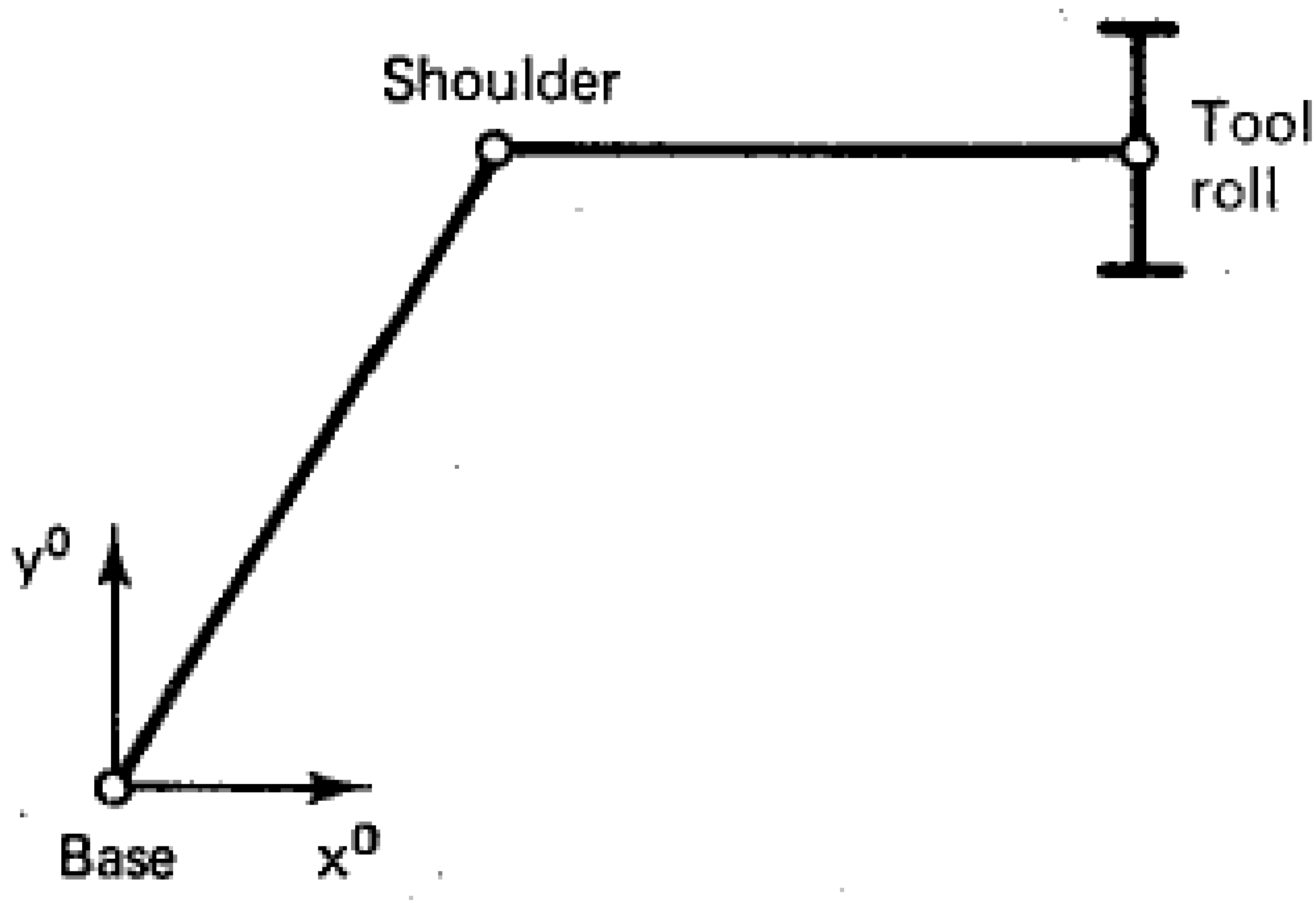
$$q_1 = \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{a_2 \sin q_2}{a_1 + a_2 \cos q_2}$$

The other way: use geometry

It may be possible to exploit considerations related to the **geometrical structure** of the manipulator

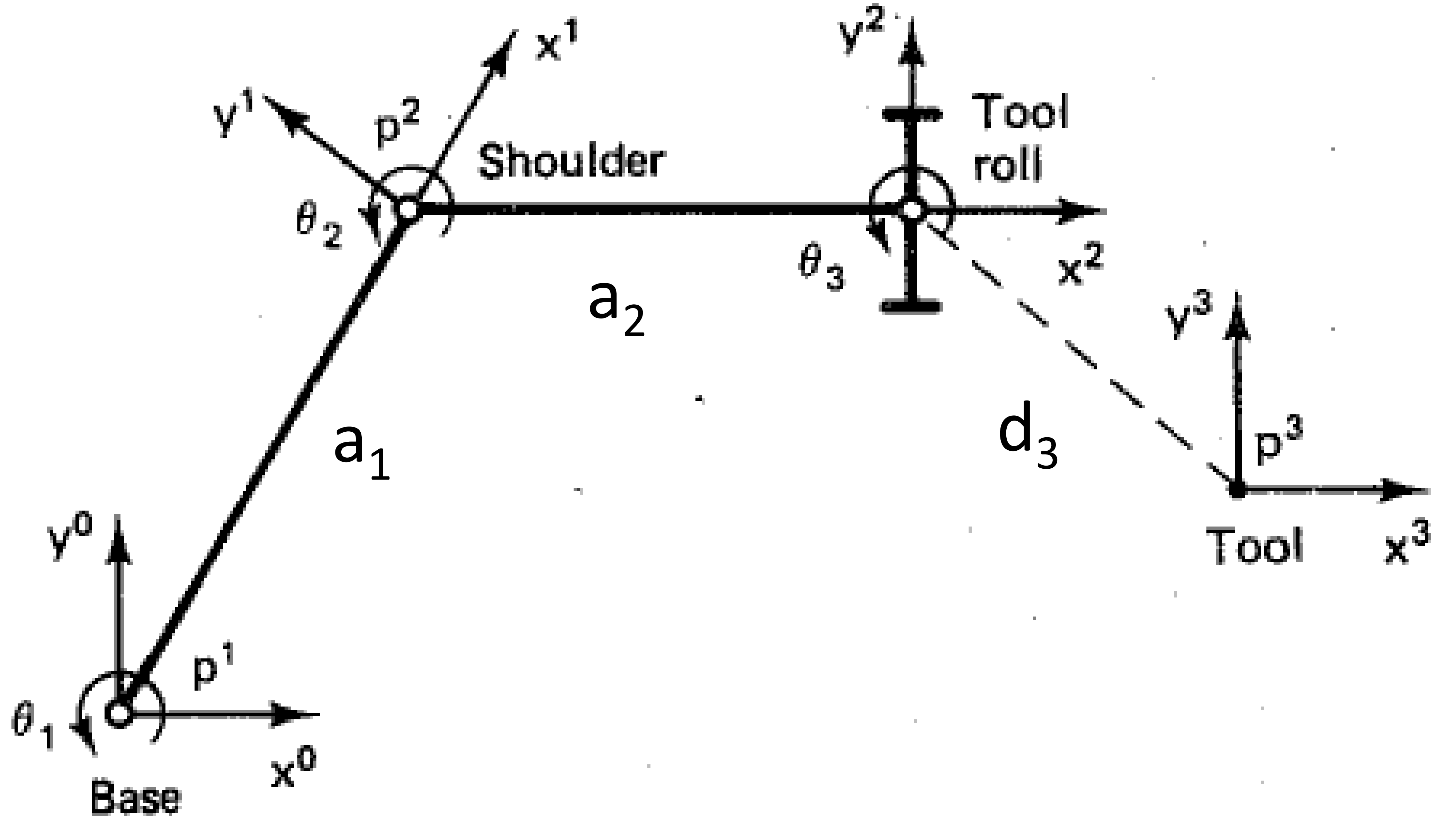
Example 2: the 3 DOF planar arm

Inverse Kinematics of a 3 DOF Planar Articulated Robot



Inverse Kinematics of a 3 DOF Planar Articulated Robot

Apply DH algorithm:



Inverse Kinematics of a 3 DOF Planar Articulated Robot

Kinematic parameters from DH algorithm:

Axis	θ	d	a	α	Home
1	q_1	0	a_1	0	$\pi/3$
2	q_2	0	a_2	0	$-\pi/3$
3	q_3	d_3	0	0	0

Articulated robot → $\boldsymbol{q} = \boldsymbol{\theta}$

Inverse Kinematics of a 3 DOF Planar Articulated Robot

Proposition 2-6-1: Link-Coordinate Transformation. Let $\{L_0, L_1, \ldots, L_n\}$ be a set of link-coordinate frames assigned by Algorithm 2-5-1, and let $[q]^k$ and $[q]^{k-1}$ be the homogeneous coordinates of a point q with respect to frames L_k and L_{k-1}, respectively. Then, for $1 \le k \le n$, we have $[q]^{k-1} = T_{k-1}^{k}[q]^k$, where:

$$T_{k-1}^{k} = \begin{bmatrix} C\theta_k & -C\alpha_k S\theta_k & S\alpha_k S\theta_k & a_k C\theta_k \\ S\theta_k & C\alpha_k C\theta_k & -S\alpha_k C\theta_k & a_k S\theta_k \\ 0 & S\alpha_k & C\alpha_k & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics of a 3 DOF Planar Articulated Robot

$$T_{\text{base}}^{\text{tool}} = T_0^1 T_1^2 T_2^3$$

$$= \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{12} & -S_{12} & 0 & a_1C_1 + a_2C_{12} \\ S_{12} & C_{12} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{123} & -S_{123} & 0 & a_1C_1 + a_2C_{12} \\ S_{123} & C_{123} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics of a 3 DOF Planar Articulated Robot

Combine tool-tip position and tool orientation into a tool-configuration vector *w*

$$w(q) = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ d_3 \\ \hline 0 \\ 0 \\ \exp(q_3/\pi) \end{bmatrix}$$

(tool-configuration)

Describing the orientation with a matrix is redundant

Definition 3-3-1: Tool-Configuration Vector. Let p and R denote the position and orientation of the tool frame relative to the base frame where q_n represents the tool roll angle. Then the *tool-configuration vector* is a vector w in $\mathbf{R}^6$ defined:

$$w \triangleq \begin{bmatrix} w^1 \\ \hline w^2 \end{bmatrix} \triangleq \begin{bmatrix} p \\ \hline [\exp(q_n/\pi)]r^3 \end{bmatrix}$$

r^3 is the third column of R

Tool roll angle q_n:

$$q_n = \pi \ln (w_4^2 + w_5^2 + w_6^2)^{1/2}$$

Inverse Kinematics of a 3 DOF Planar Articulated Robot

$$q_2 = \pm \arccos \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

↓

$$q_1 = \operatorname{atan2}\left[(a_1 - a_2 C_2)w_1 + a_2 S_2 w_2,\ (a_1 + a_2 C_2)w_2 - a_2 S_2 w_1\right]$$

↓

$$q_3 = \pi \ln w_6$$

Is a generalization possible?

Yes! The PIEPER APPROACH (1968)

Many industrial manipulators have a kinematically decoupled structure, for which it is possible to decompose the problem into two (simpler) sub-problems:

1) Inverse kinematics for the position

$$\mathbf{p} = (x, y, z) \rightarrow q_1, q_2, q_3$$

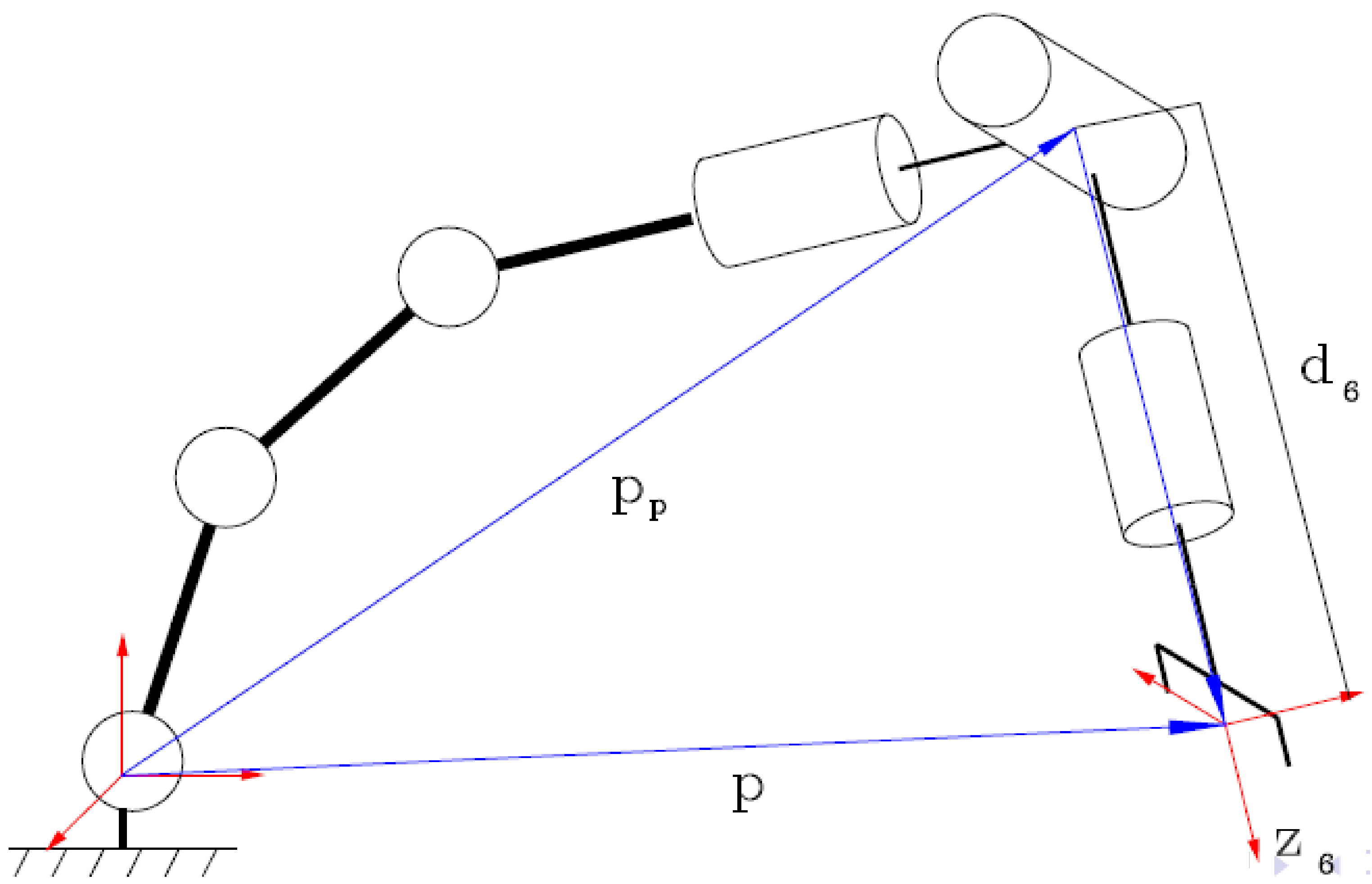
2) Inverse kinematics for the orientation

$$\mathbf{R} \rightarrow q_4, q_5, q_6.$$

The Pieper Approach

Given a 6 DOF manipulator, a sufficient condition to find a closed form solution for the IK problem is that the kinematic structure presents:

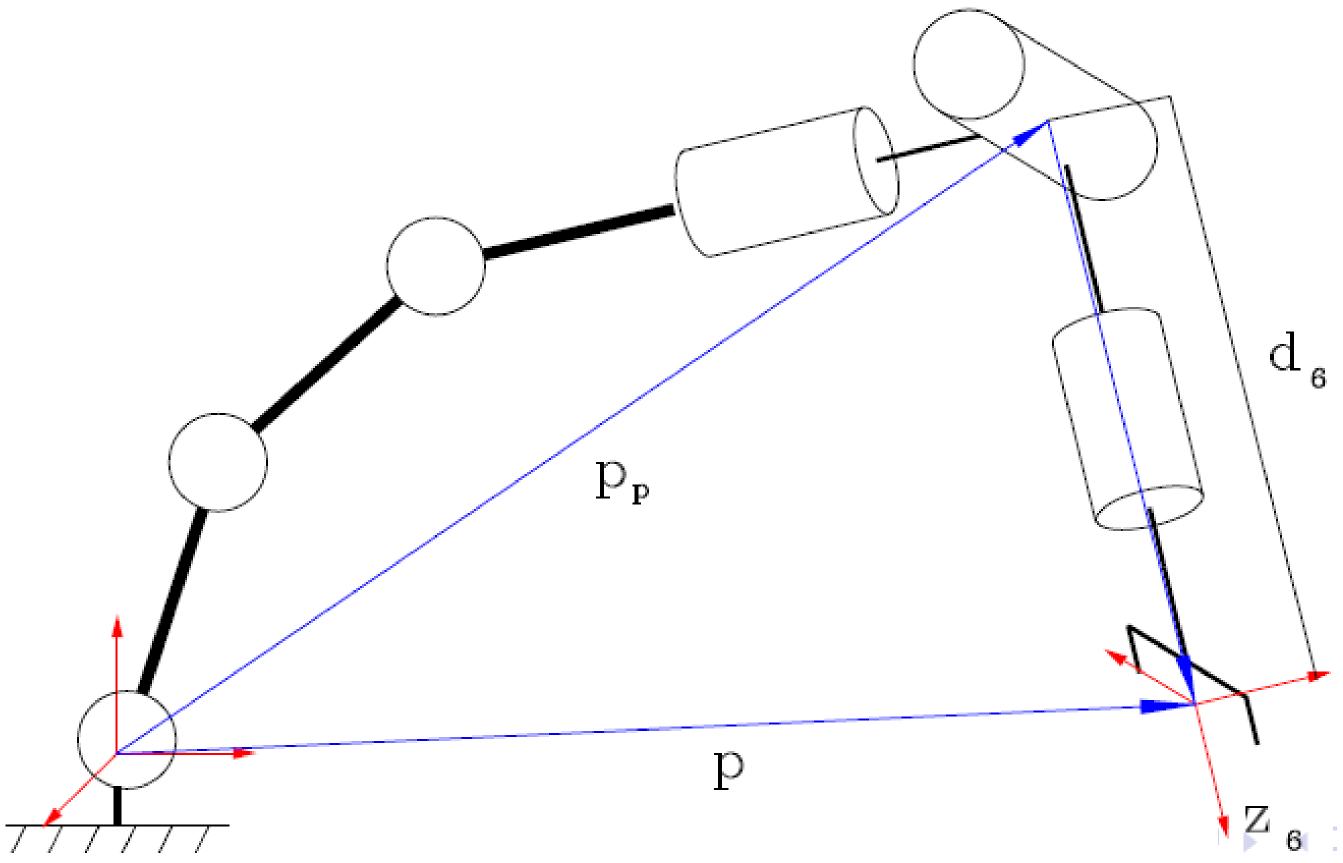
1. three consecutive rotational joints with axes intersecting in a single point, or
2. three consecutive rotational joints with parallel axes.



The Pieper Approach

In many 6 DOF industrial manipulators, the first 3 DOF are usually devoted to position the wrist, that has 3 additional DOF give the correct orientation to the end-effector.

In these cases, it is quite simple to decompose the IK problem in the two subproblems (position and orientation).



The Pieper Approach

In case of a manipulator with a spherical wrist, a natural choice is to decompose the problem in

A. IK for the point $\mathbf{p}_p$ (center of the spherical wrist)

B. solution of the orientation IK problem

Therefore:

1) The point $\mathbf{p}_p$ is computed since T_0^6 is known (submatrices $\mathbf{R}$ and $\mathbf{p}$): $\mathbf{p}_p = \mathbf{p} - d_6\mathbf{a}$

   $\mathbf{p}_p$ depends only on the joint variables (q_1, q_2, q_3);

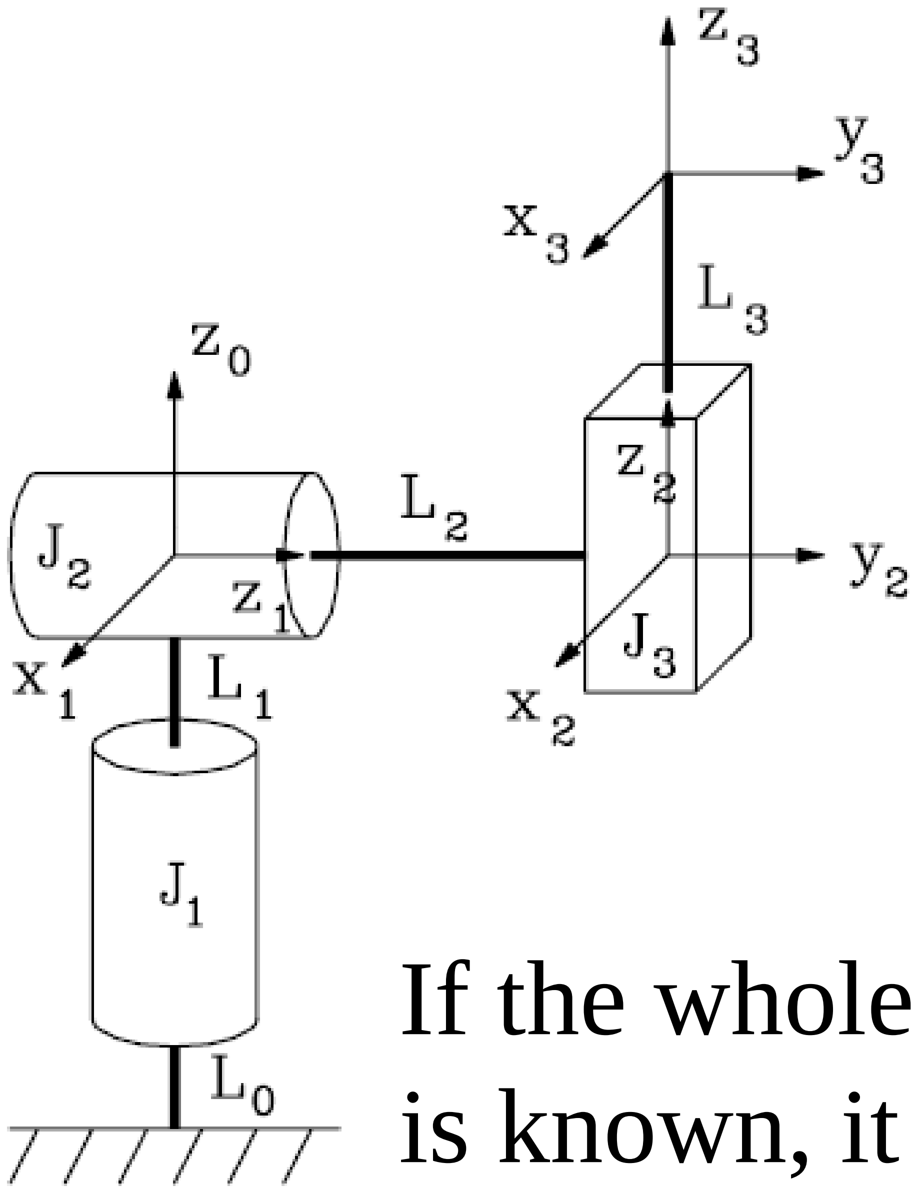
2) The IK problem is solved for (q_1, q_2, q_3);

3) The rotation matrix $\mathbf{R}_0^3$ related to the first three joints is computed;

4) The matrix $\mathbf{R}_3^6 = \left(\mathbf{R}_0^3\right)^T \mathbf{R}$ is computed;

5) The IK problem for the rotational part is solved (Euler)

Solution of the spherical manipulator [1]



Direct kinematic model:

$$T_0^3 = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

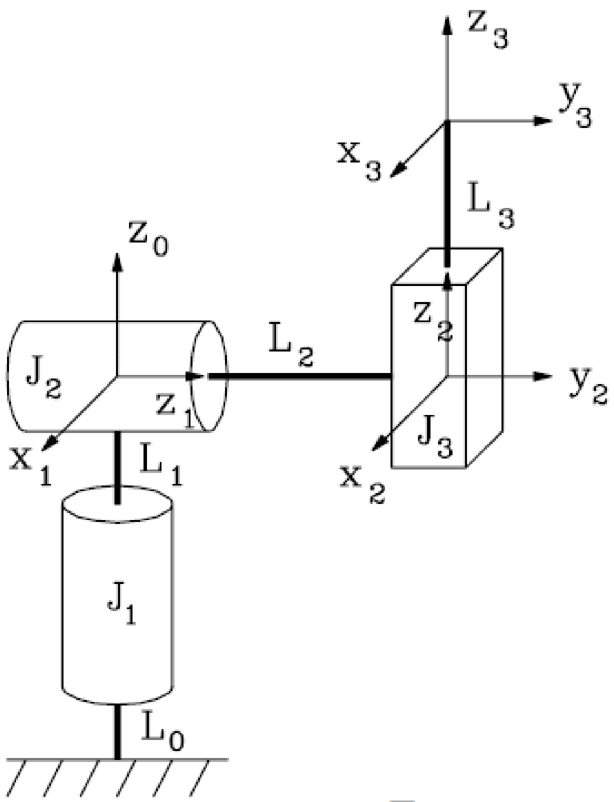
$$\begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 & -d_2 S_1 + d_3 C_1 S_2 \\ C_2 S_1 & C_1 & S_1 S_2 & d_2 C_1 + d_3 S_1 S_2 \\ -S_2 & 0 & C_2 & d_3 C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If the whole matrix T_0^3 is known, it is possible to compute:

$$\begin{cases} \theta_1 & = & \text{atan2}\,(-s_x, s_y) \\ \theta_2 & = & \text{atan2}\,(-n_z, a_z) \\ d_3 & = & p_z / \cos\theta_2 \end{cases}$$

Unfortunately, according to the Pieper approach, only **p** is known!

Solution of the spherical manipulator [2]



We know only the position vector **p**

We have

$$\left[T_0^1\right]^{-1} T_0^3 = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 & d_3 S_2 \\ S_2 & 0 & -C_2 & -d_3 C_2 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= T_1^2 T_2^3$$

Solution of the spherical manipulator [3]

By equating the position vectors

$$ {}^1\mathbf{p}_p = \begin{bmatrix} p_x C_1 + p_y S_1 \\ -p_z \\ -p_x S_1 + p_y C_1 \end{bmatrix} = \begin{bmatrix} d_3 S_2 \\ -d_3 C_2 \\ d_2 \end{bmatrix} $$

The vector ${}^1\mathbf{p}_p$ depends only on θ_2 and d_3. Let's define $a = \tan\theta_1/2$

Then

$$ C_1 = \frac{1-a^2}{1+a^2} \qquad S_1 = \frac{2a}{1+a^2} $$

By substitution in the last element of ${}^1\mathbf{p}_p$

$$ (d_2 + p_y)a^2 + 2p_x a + d_2 - p_y = 0 \qquad \Longrightarrow \qquad a = \frac{-p_x \pm \sqrt{p_x^2 + p_y^2 - d_2^2}}{d_2 + p_y} $$

Two possible solutions!

of course: $((p_x^2 + p_y^2 - d_2^2) > 0??)$

Then

$$ \theta_1 = 2\ \text{atan2}\left(-p_x \pm \sqrt{p_x^2 + p_y^2 - d_2^2},\ \ d_2 + p_y\right) $$

Solution of the spherical manipulator [4]

Since

$$ {}^{1}\mathbf{p}_p = \begin{bmatrix} p_x C_1 + p_y S_1 \\ -p_z \\ -p_x S_1 + p_y C_1 \end{bmatrix} = \begin{bmatrix} d_3 S_2 \\ -d_3 C_2 \\ d_2 \end{bmatrix} $$

From the first two elements $$ \frac{p_x C_1 + p_y S_1}{-p_z} = \frac{d_3 S_2}{-d_3 C_2} $$

from which $$ \theta_2 = \mathrm{atan2}\left(p_x C_1 + p_y S_1,\ p_z\right) $$

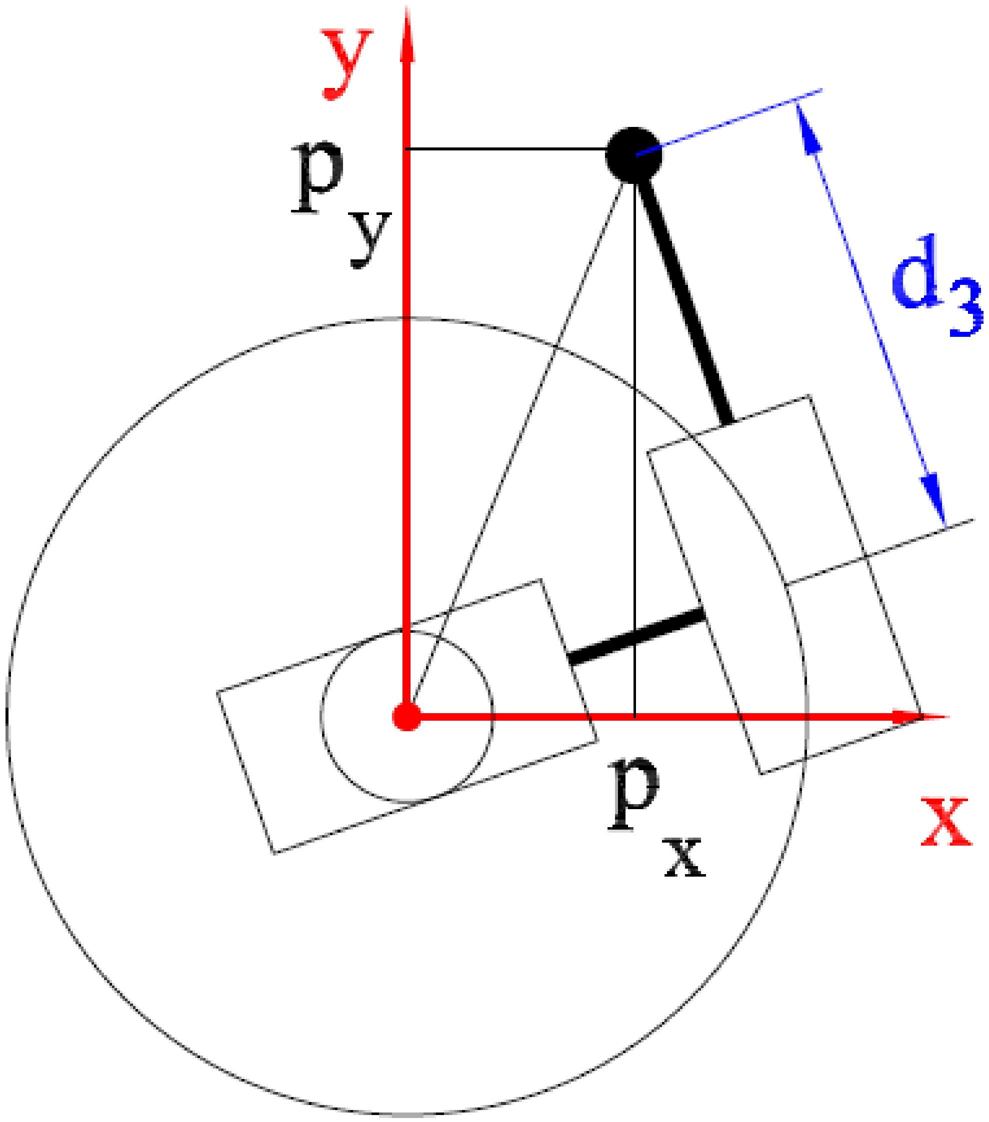
Finally, if the first two elements are squared and added together

$$ d_3 = \sqrt{(p_x C_1 + p_y S_1)^2 + p_z^2} $$

Solution of the spherical manipulator [5]

Note that two possible solutions exist considering the position of the end-effector (wrist) only. If also the orientation is considered, the solution (if it exists) is unique.

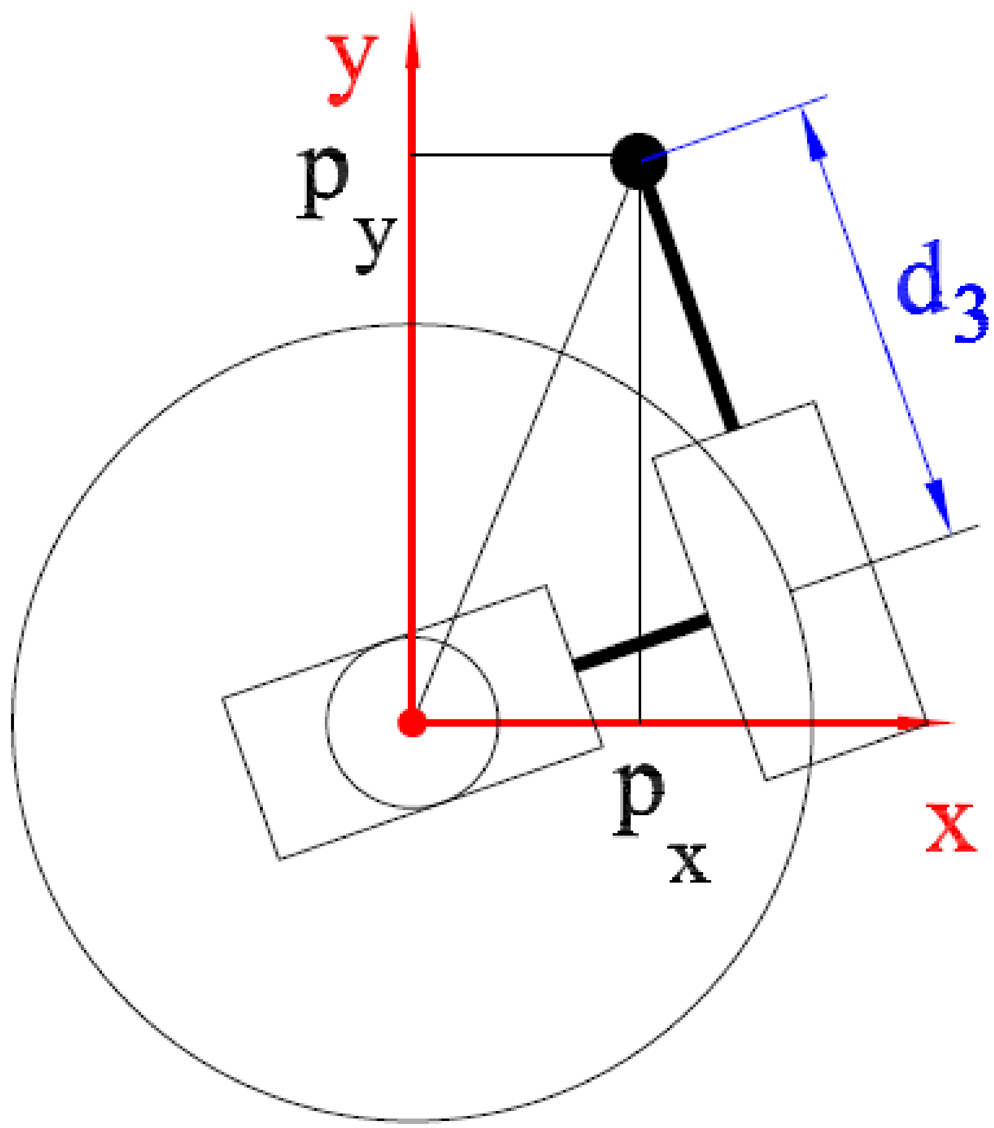
We have seen that the relation $(p_x^2 + p_y^2 - d_2^2) > 0$ must hold:



Solution of the spherical manipulator [5]

Note that two possible solutions exist considering the position of the end-effector (wrist) only. If also the orientation is considered, the solution (if it exists) is unique.

We have seen that the relation $(p_x^2 + p_y^2 - d_2^2) > 0$ must hold:



Solution of the spherical manipulator [6]

Numerical example: Given a spherical manipulator with $d_2 = 0.8$ m in the pose

$$\theta_1 = 20^o, \qquad \theta_2 = 30^o, \qquad d_3 = 0.5 \ m$$

We have:

$$T_0^3 = \left[\begin{array}{ccc|c} 0.8138 & -0.342 & 0.4698 & -0.0387 \\ 0.2962 & 0.9397 & 0.171 & 0.8373 \\ -0.5 & 0 & 0.866 & 0.433 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The solution based on the whole matrix T_0^3 is:

$$\theta_1 = 20^o, \ \theta_2 = 30^o, \ d_3 = 0.5.$$

The solution based on the vector **p** gives:

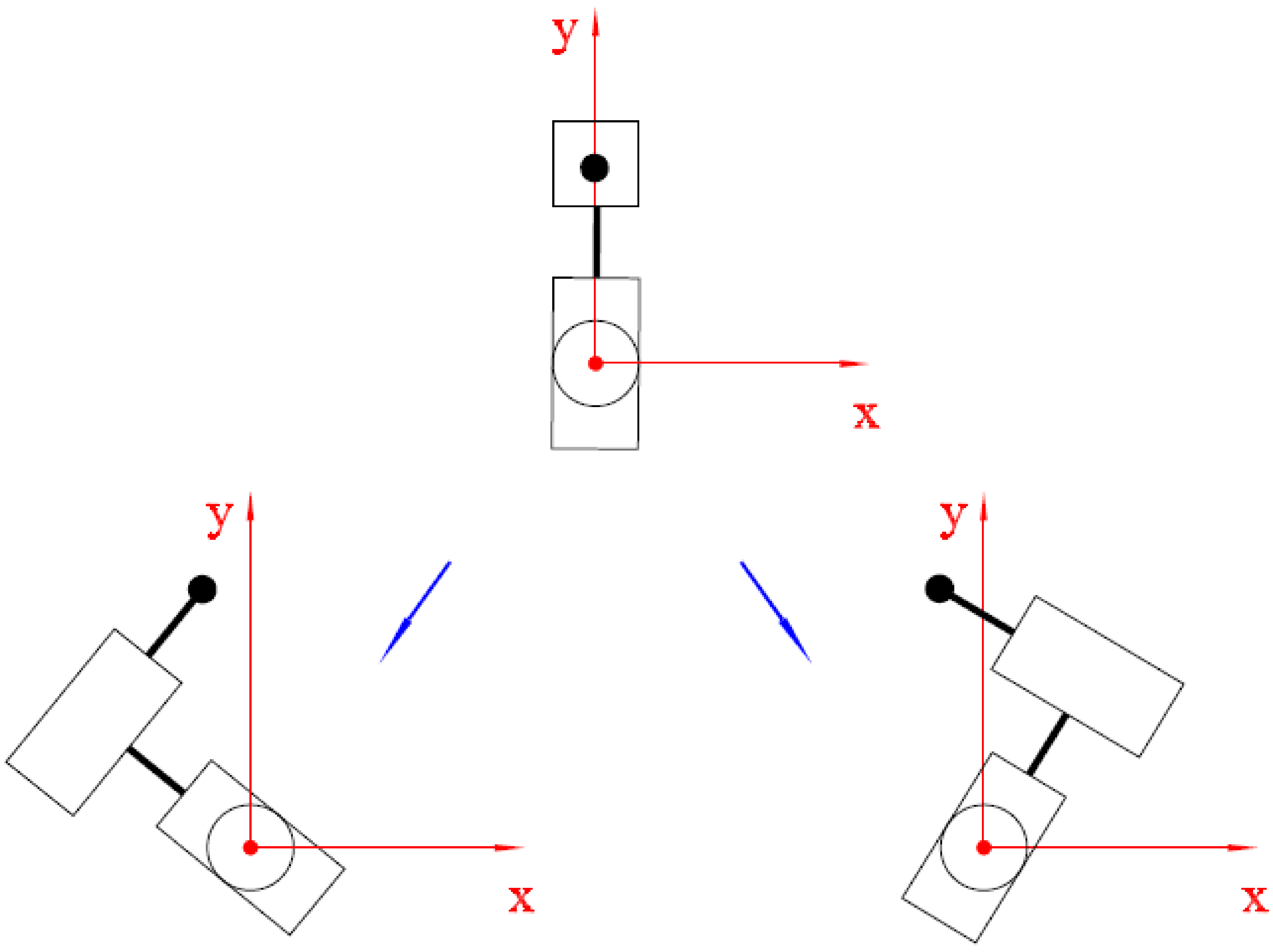
a) $\theta_1 = 20^o, \ \theta_2 = 30^o, \ d_3 = 0.5$ (with the "+" sign).

b) $\theta_1 = -14.7^o, \ \theta_2 = -30^o, \ d_3 = 0.5$ (with the "-" sign).

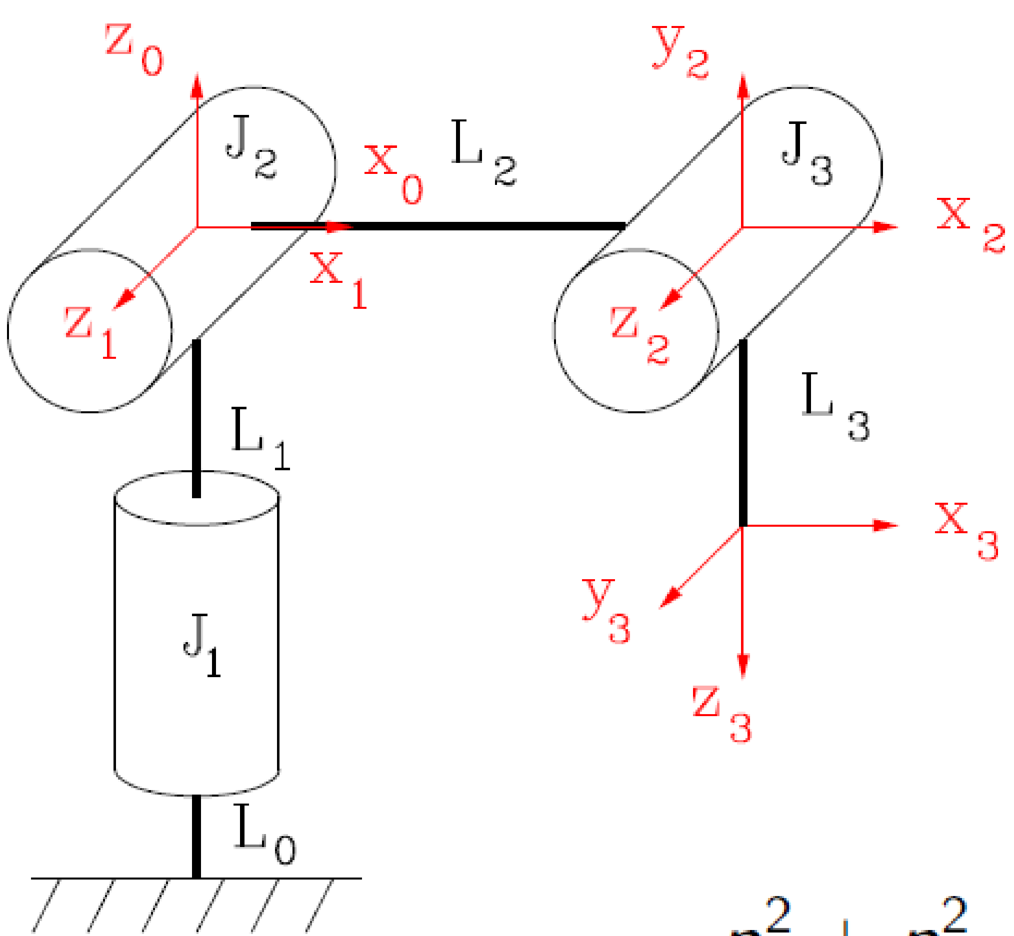
Solution of the spherical manipulator [7]

a) $\theta_1 = 20^o,\ \theta_2 = 30^o,\ d_3 = 0.5$ (with the "+" sign).

b) $\theta_1 = -14.7^o,\ \theta_2 = -30^o,\ d_3 = 0.5$ (with the "-" sign).



Solution of the 3 DOF anthropomorphic arm [1]



From the kinematic structure, one obtains

$$\theta_1 = \text{atan2}\,(p_y,\ p_x)$$

Concerning θ_2 and θ_3, note that the arm moves in a plane defined by θ_1 only

We obtain $$C_3 = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2}{2a_2 a_3}$$ $$S_3 = \pm\sqrt{1 - C_3^2}$$

$$\theta_3 = \text{atan2}\,(S_3,\ C_3)$$

Moreover, by geometrical arguments, it is possible to state that:

$$\theta_2 = \text{atan2}\,(p_z,\ \sqrt{p_x^2 + p_y^2}\ ) - \text{atan2}\,(a_3 S_3,\ a_2 + a_3 C_3)$$

Solution of the 3 DOF anthropomorphic arm [2]

Note that also the following solution is valid

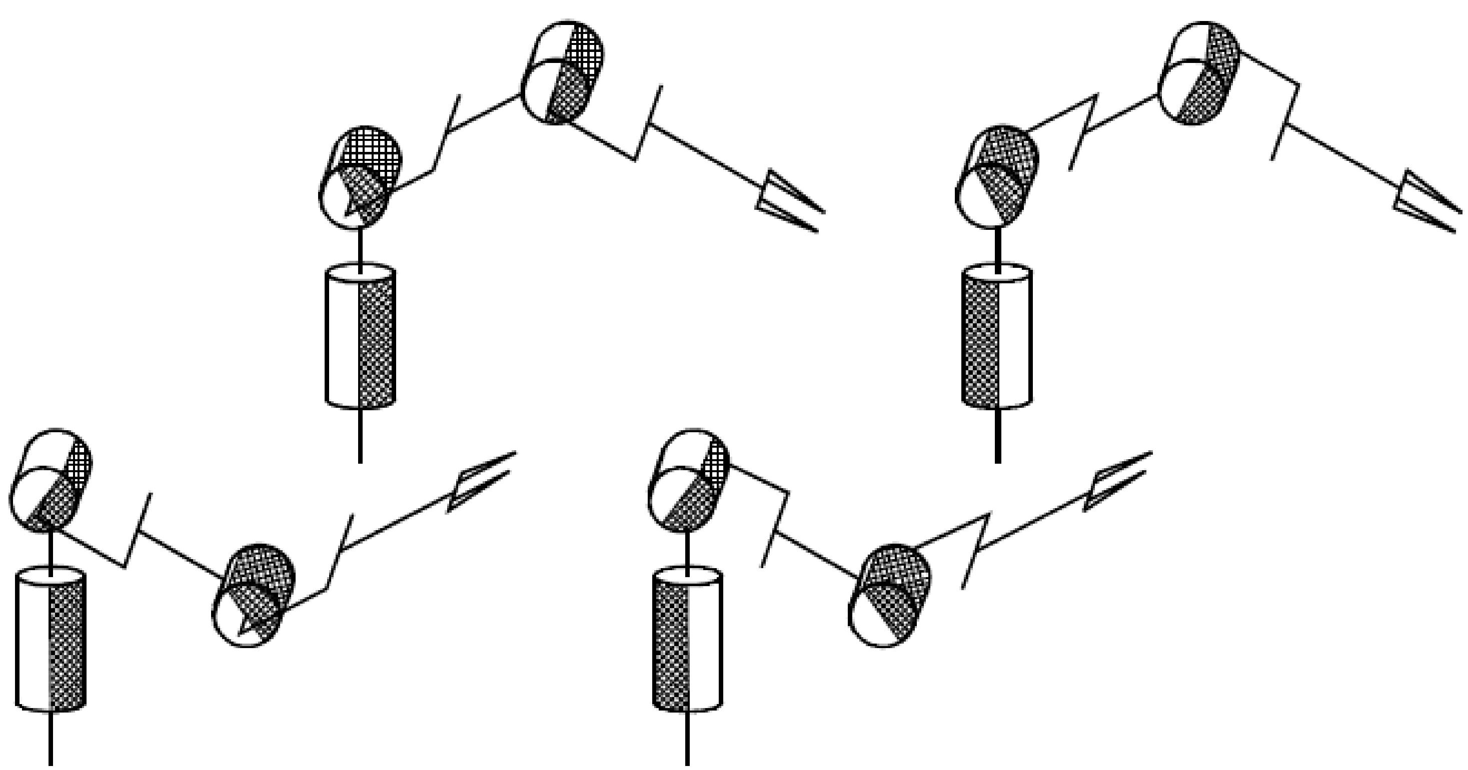
$$\theta_1 = \pi + \text{atan2}(p_y,\ p_x), \qquad \theta_2' = \pi - \theta_2$$

Then, FOUR possible solutions exist:

shoulder right - elbow up; shoulder right - elbow down;

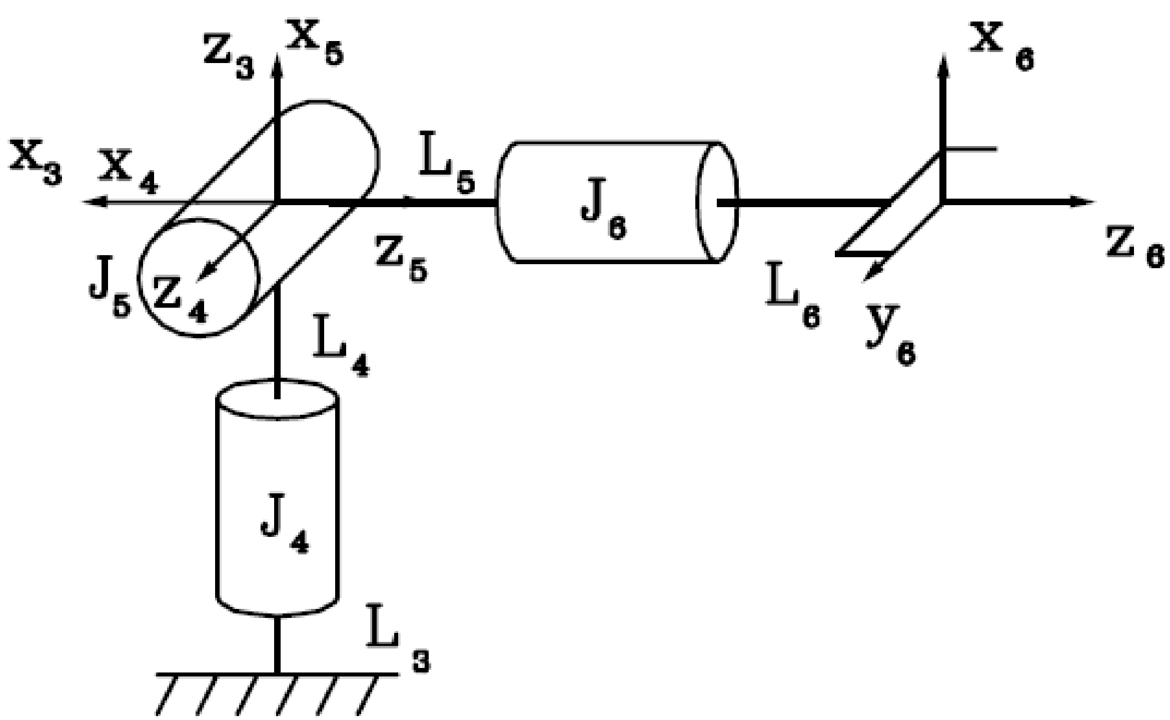
shoulder left - elbow up; shoulder left - elbow down;

Same position, but different orientation!



Note that the conditions $p_x \neq 0$, $p_y \neq 0$ must hold (o.w. singular configuration)

Solution of the spherical wrist [1]



Let us consider the matrix

$$^{3}\mathbf{R}_6 = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}$$

From the direct kinematic equations one obtains

$$^{3}\mathbf{R}_6 = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -S_4C_6 - C_4C_5S_6 & C_4S_5 \\ S_4C_5C_6 + C_4S_6 & C_4C_6 - S_4C_5S_6 & S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 \end{bmatrix}$$

Solution of the spherical wrist [2]

$$
{}^{3}\mathbf{R}_6 = \begin{bmatrix} C_4 C_5 C_6 - S_4 S_6 & -S_4 C_6 - C_4 C_5 S_6 & C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 & C_4 C_6 - S_4 C_5 S_6 & S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 \end{bmatrix}
$$

The solution is then computed as (ZYZ Euler angles):

- $\theta_5 \in [0, \pi]$:

$$
\begin{aligned}
\theta_4 &= \operatorname{atan2}(a_y,\ a_x) \\
\theta_5 &= \operatorname{atan2}(\sqrt{a_x^2 + a_y^2},\ \ a_z) \\
\theta_6 &= \operatorname{atan2}(s_z,\ -n_z)
\end{aligned}
$$

- $\theta_5 \in [-\pi, 0]$:

$$
\begin{aligned}
\theta_4 &= \operatorname{atan2}(-a_y,\ -a_x) \\
\theta_5 &= \operatorname{atan2}(-\sqrt{a_x^2 + a_y^2},\ \ a_z) \\
\theta_6 &= \operatorname{atan2}(-s_z,\ n_z)
\end{aligned}
$$

Solution of the spherical wrist [3]

Numerical example: Let us consider a spherical wrist in the pose

$$\theta_4 = 10^o \qquad \theta_5 = 20^o, \qquad \theta_6 = 30^o$$

Then

$$^3\mathbf{R}_6 = \begin{bmatrix} 0.7146 & -0.6131 & 0.3368 \\ 0.6337 & 0.7713 & 0.0594 \\ -0.2962 & 0.1710 & 0.9397 \end{bmatrix}$$

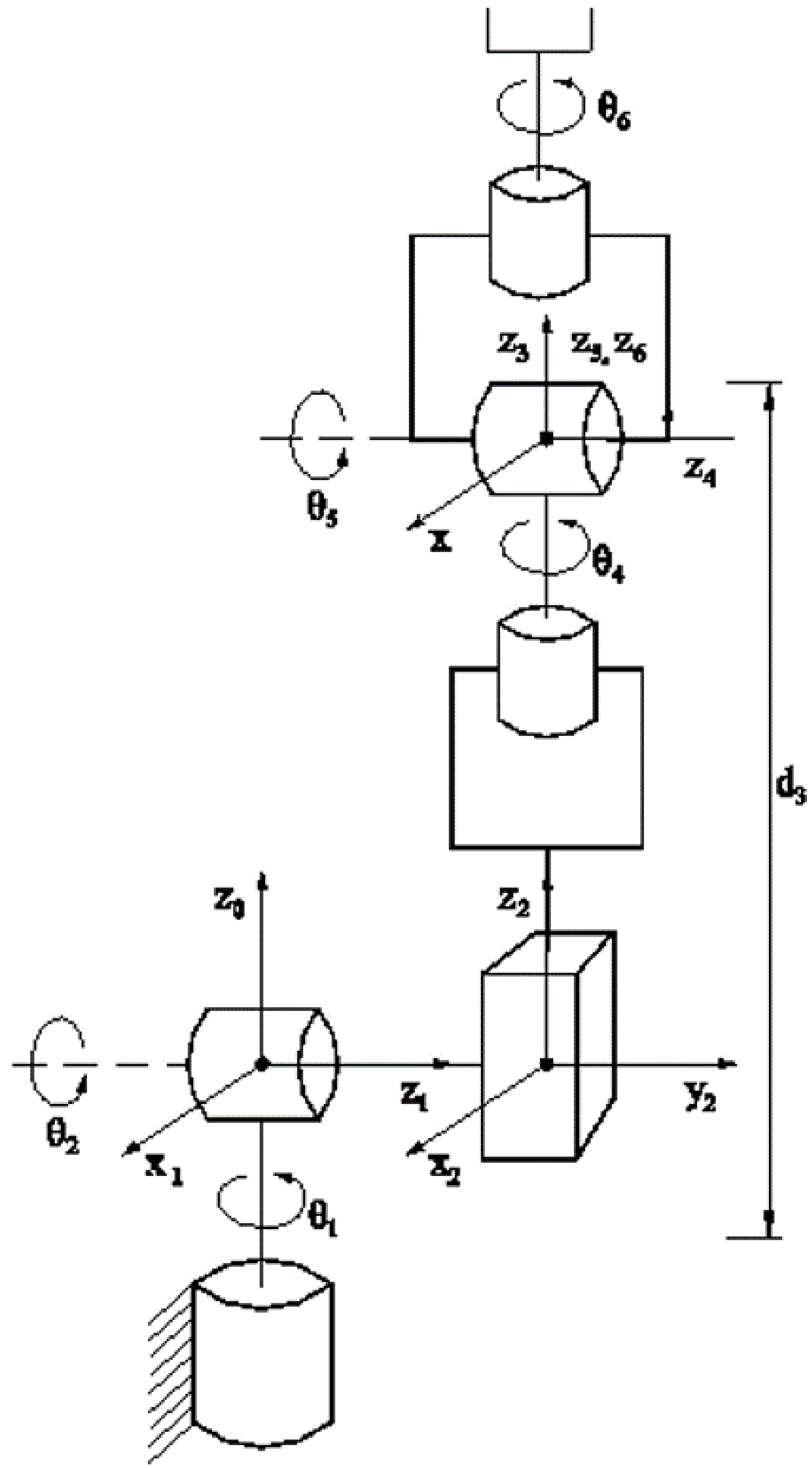
Therefore, if

- $\theta_5 \in [0, \ \pi]$ $\qquad \theta_4 = 10^o \qquad \theta_5 = 20^o, \qquad \theta_6 = 30^o$
- $\theta_5 \in [-\pi, \ 0]$ $\qquad \theta_4 = -170^o \qquad \theta_5 = -20^o, \qquad \theta_6 = -30^o$

- Note that the PUMA has an anthropomorphic structure (4 solutions) and a spherical wrist (2 solutions):

$\Rightarrow$ 8 different configurations are possible!

STANFORD MANIPULATOR



Stanford manipulator IK [1]

We need the forward kinematics:

$$u_x = c_1[c_2(c_6c_4c_5 - s_4s_6) - s_5s_2c_6] - s_1[c_6c_5s_4 + c_4s_6]$$

$$u_y = s_1[c_2(c_6c_4c_5 - s_4s_6) - s_5s_2c_6] + c_1[c_6c_5s_4 + c_4s_6]$$

$$u_z = -s_2c_6c_4c_5 - s_5c_2c_6 + s_6s_2s_4$$

$$v_x = c_1[-c_2(s_6c_4c_5 + s_4c_6) + s_5s_2s_6] - s_1[-s_6c_5s_4 + c_4c_6]$$

$$v_y = s_1[-c_2(s_6c_4c_5 + s_4c_6) + s_5s_2s_6] + c_1[-s_6c_5s_4 + c_4c_6]$$

$$v_z = s_2s_6c_4c_5 + s_5c_2s_6 + c_6s_2s_4$$

$$w_x = c_1[c_2c_4s_5 + c_5s_2] - s_1s_4s_5$$

$$w_y = s_1[c_2c_4s_5 + c_5s_2] + c_1s_4s_5$$

$$w_z = -s_2c_4s_5 + c_5c_2$$

$$p_x = c_1s_2d_3 - s_1d_2$$

$$p_y = s_1s_2d_3 + c_1d_2$$

$$p_z = d_3c_2$$

Stanford manipulator IK [2]

$$\theta_1 = \mathrm{Tan}^{-1}\left(\frac{p_y}{p_x}\right) - \mathrm{Tan}^{-1}\left(\frac{d_2}{+/-\sqrt{r^2 - d_2^{\;2}}}\right) \qquad r = \sqrt{p_x^{\;2} + p_y^{\;2}}$$

$$\theta_2 = \mathrm{Tan}^{-1}\left(\frac{c_1 p_x + s_1 p_y}{p_z}\right)$$

$$d_3 = (p_x c_1 + s_1 p_y)\, s_2 + p_z c_2$$

$$\theta_4 = \mathrm{Tan}^{-1}\left[\frac{-s_1 w_x + c_1 w_y}{c_2(c_1 w_x + s_1 w_y) - s_2 w_z}\right]$$

$$\theta_5 = \mathrm{Tan}^{-1}\left(\frac{c_4[c_2(c_1 w_x + s_1 w_y) - s_2 w_z] + s_4(c_1 w_y - s_1 w_x)}{s_2(c_1 w_x + s_1 w_y) + c_2 w_z}\right)$$

$$\theta_6 = \mathrm{Tan}^{-1}\left[\frac{-c_5[c_4(c_2 I - s_2 v_z) + s_4 n] + s_5(s_2 I + c_2 v_z)}{-s_4(c_2 I - s_2 v_z) + c_4 n}\right]$$

$$I = c_1 v_x + s_1 v_y, \quad n = -s_1 v_x + c_1 v_y$$

The Inverse Kinematics Problem

Search “around” for your robot of interest!!
(or part of robot)

The secret: Use well known robots!!

(it was mentioned in the introduction: “…for the most common kinematic structures, a scheme for obtaining the solution has been found”)