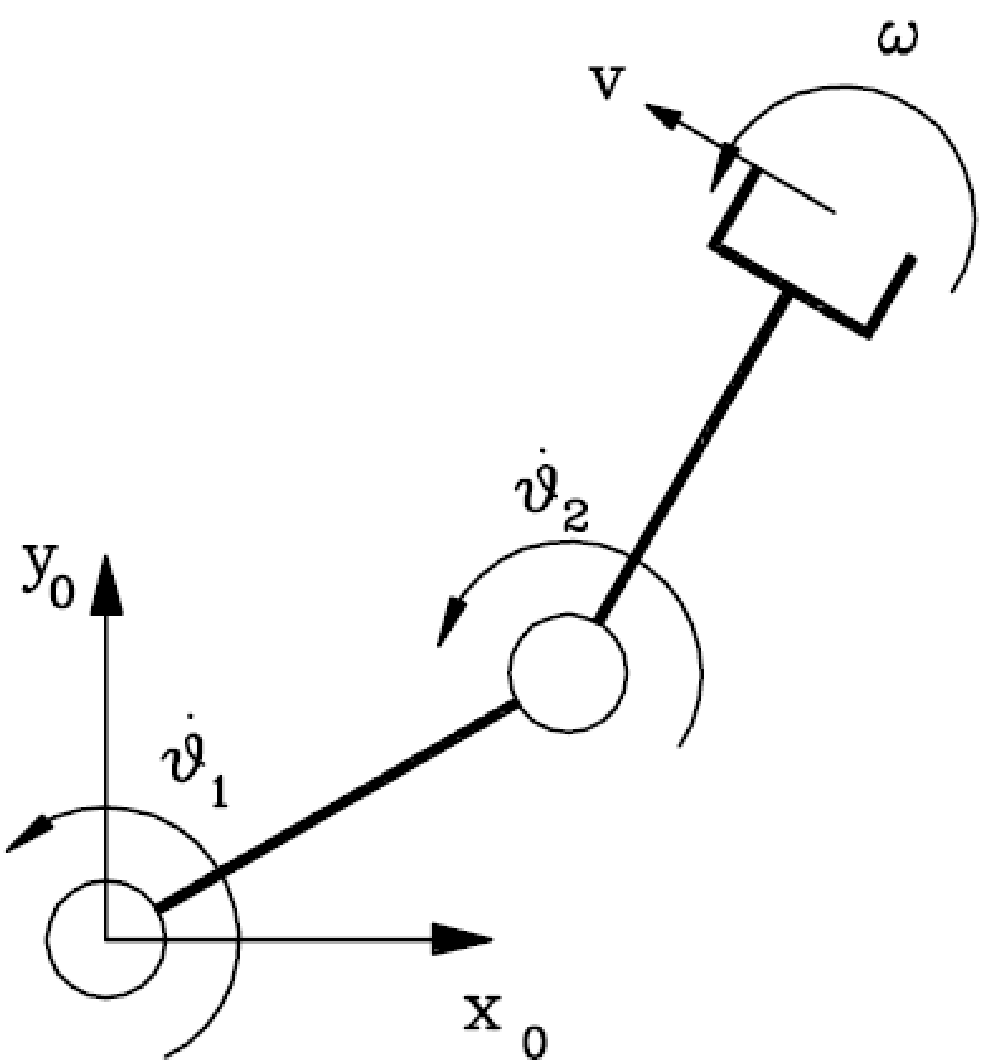
Differential Kinematics

- Relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)

- Instantaneous velocity mappings can be obtained through time derivation of the direct kinematics or in a geometric way, directly at the differential level

- Different treatments arise for rotational quantities
  - ❑ establish the link between angular velocity and time derivative of a rotation matrix
  - ❑ establish the link between angular velocity and time derivative of the angles in a minimal representation of orientation

Differential Kinematics: the Jacobian matrix



In robotics it is of interest to define, besides the mapping between the joint and workspace position and orientation (i.e. the kinematic equations), also:

- The relationship between the joints and end-effector **velocities**:

$$\begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} \Longleftrightarrow \dot{\mathbf{q}}$$

- The relationship between the force applied on the environment by the manipulator and the corresponding joint torques

$$\begin{bmatrix} \mathbf{f} \\ \mathbf{n} \end{bmatrix} \Longleftrightarrow \boldsymbol{\tau}$$

These two relationships are based on a linear operator, a matrix J, called the Jacobian of the manipulator.

Differential Kinematics: the Jacobian matrix

$$\dot{w} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$ **Cartesian Velocity**

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$ **Joint Velocity**

$$\dot{w} = J\dot{q}$$

If it 'exists' we can define the Inverse Jacobian as:

$$\dot{q} = J^{-1}\dot{w}$$

- ❖ The Jacobian is a mapping tool that relates Cartesian velocities (of the *n* frame) to the movement of the individual robot joints
- ❖ The Jacobian collectively represents the sensitivities of individual end-effector coordinates to individual joint displacements

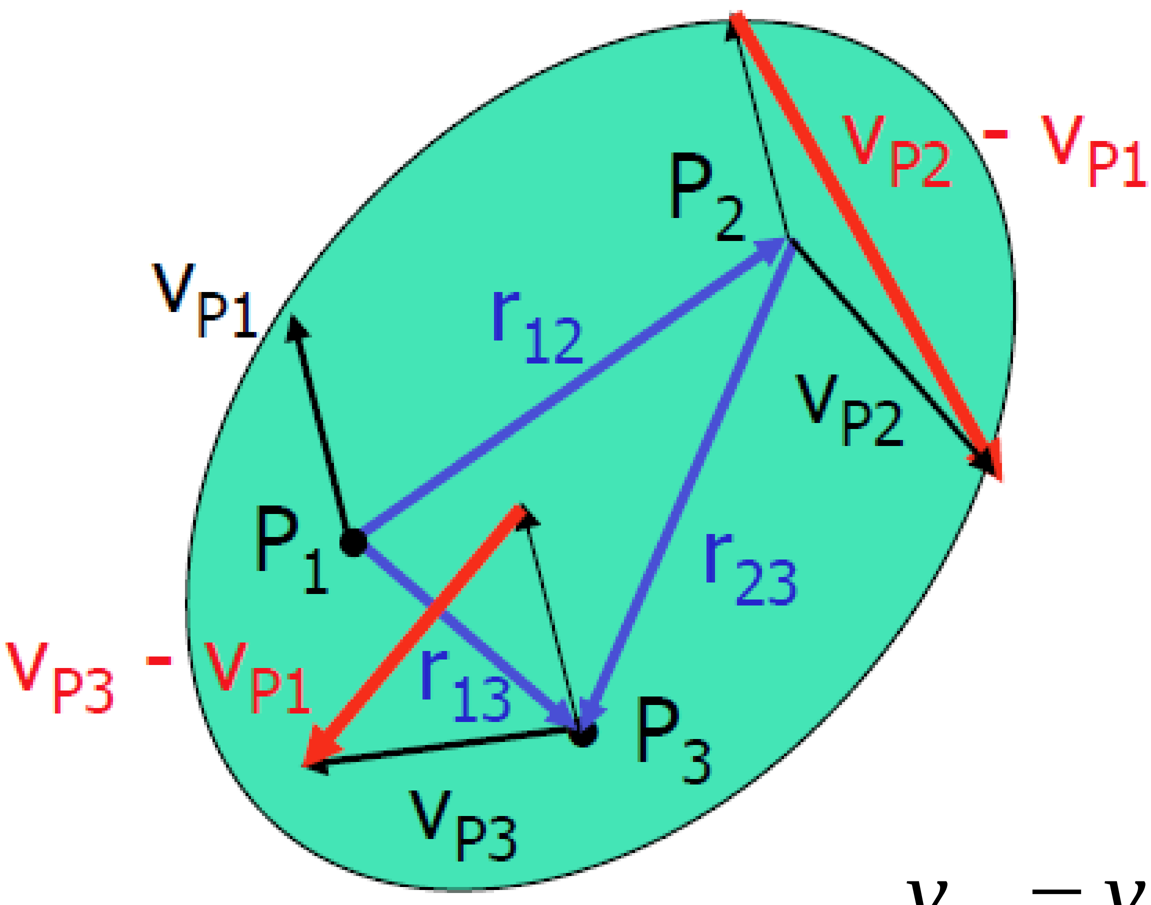
The Jacobian matrix

In robotics, the Jacobian is used for several purposes:

- ❑ To define the relationship between joint and workspace velocities
- ❑ To define the relationship between forces/torques between the spaces
- ❑ To study the singular configurations
- ❑ To define numerical procedures for the solution of the IK problem
- ❑ To study the manipulability properties

(Angular velocity of a rigid body)

"rigidity" constraint on distances among points: $||r_{ij}||$ = constant



$v_{Pi} - v_{Pj}$ orthogonal to r_{ij}

$$v_{P2} - v_{P1} = \omega_1 \times r_{12} \quad \text{(A)}$$

$$v_{P3} - v_{P1} = \omega_1 \times r_{13} \quad \text{(B)}$$

$$v_{P3} - v_{P2} = \omega_2 \times r_{23} \quad \text{(C)}$$

$\forall$P1, P2, P3: (B)−(A) = (C) $\Rightarrow$

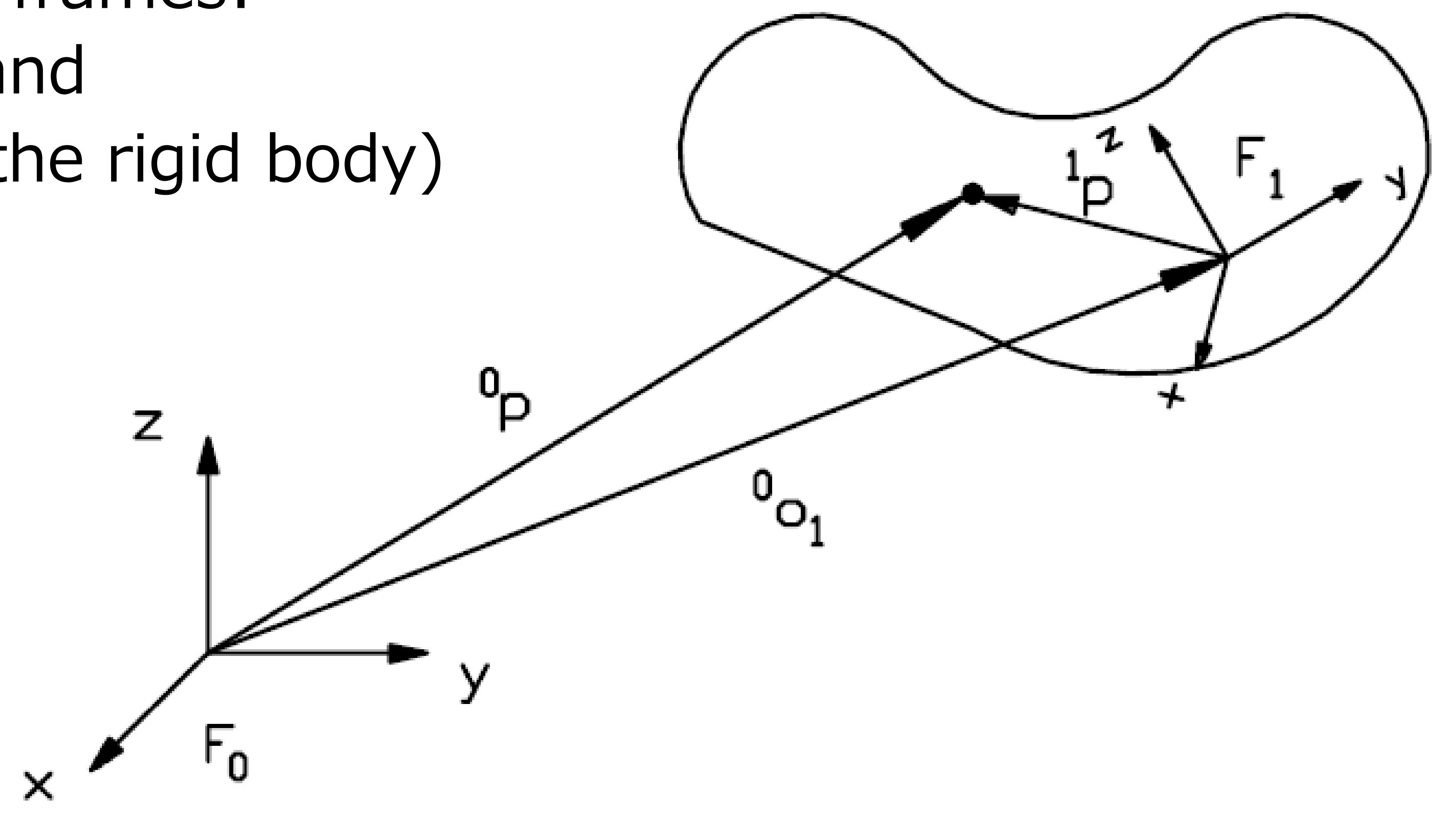
$$\omega_1 = \omega_2 = \omega$$

$$v_{Pj} = v_{Pi} + \omega \times r_{ij} = v_{Pi} + S(\omega)\, r_{ij} \Leftrightarrow \dot{r}_{ij} = \omega \times r_{ij}$$

- the angular velocity ω is associated to the whole body (**not** to a point)
- if $\exists$ P1, P2 with $v_{P1} = v_{P2} = 0$: **pure rotation** (circular motion of all $P_j \notin$ line P_1P_2)
- $\omega = 0$: **pure translation** (**all** points have the same velocity v_P)

Velocity domain

- The translational and rotational velocities are considered separately
- Let us consider two frames:
  - $\mathcal{F}_0$ (base frame) and
  - $\mathcal{F}_1$ (integral with the rigid body)



- The **translational velocity** of point **p** of the rigid body, with respect to $\mathcal{F}_0$, is defined as the derivative (w.r.t time) of **p**, denoted as $\dot{\mathbf{p}}$:

$$\dot{\mathbf{p}} = \frac{d\mathbf{p}}{dt}$$

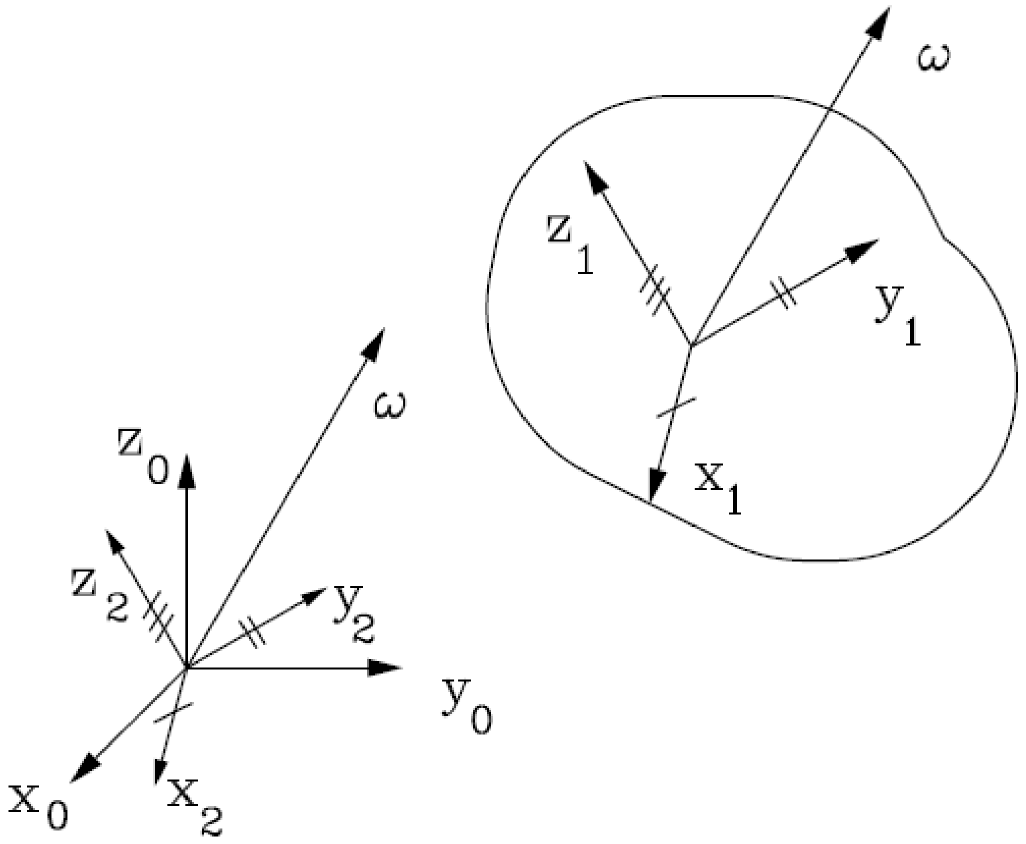
Velocity domain

For **the rotational velocity**, two different definitions are possible:

- A triplet $\gamma \in \mathbb{R}^3$ giving the orientation of $\mathcal{F}_1$ with respect to $\mathcal{F}_0$ (Euler, RPY,... angles) is adopted, and its derivative is used to define the rotational velocity $\dot{\gamma}$:

$$\dot{\gamma} = \frac{d\gamma}{dt}$$

- An angular velocity vector ω is defined, giving the rotational velocity of a third frame $\mathcal{F}_2$ with origin coincident with $\mathcal{F}_0$ and axes parallel to $\mathcal{F}_1$



The velocity vector ω is placed in the origin, and its direction coincides with the instantaneous rotation axis of the rigid body

Jacobian: Analytical and Geometrical expressions

- The two descriptions lead to different results concerning the expression of the Jacobian matrix, in particular in the part relative to the rotational velocity
- One obtains (respectively) the:

❑ Analytic Jacobian J_A

The end-effector pose is expressed with reference to a minimal representation in the operational space; then, we can compute the Jacobian matrix via differentiation of the direct kinematics function w.r.t. the joint variables

❑ Geometric Jacobian J_G

The relationship between the joint velocities and the corresponding end-effector linear and angular velocity

These two expressions are different (in general)!

Two problems

Problem 1: Integration of the rotational velocity ω

$$\int \dot{\gamma} dt \rightarrow \gamma \text{ (orientation of the rigid body)}$$

$$\int \omega dt \rightarrow ??$$

Example: Let's consider a rigid body and the following rotational velocities

Case a)

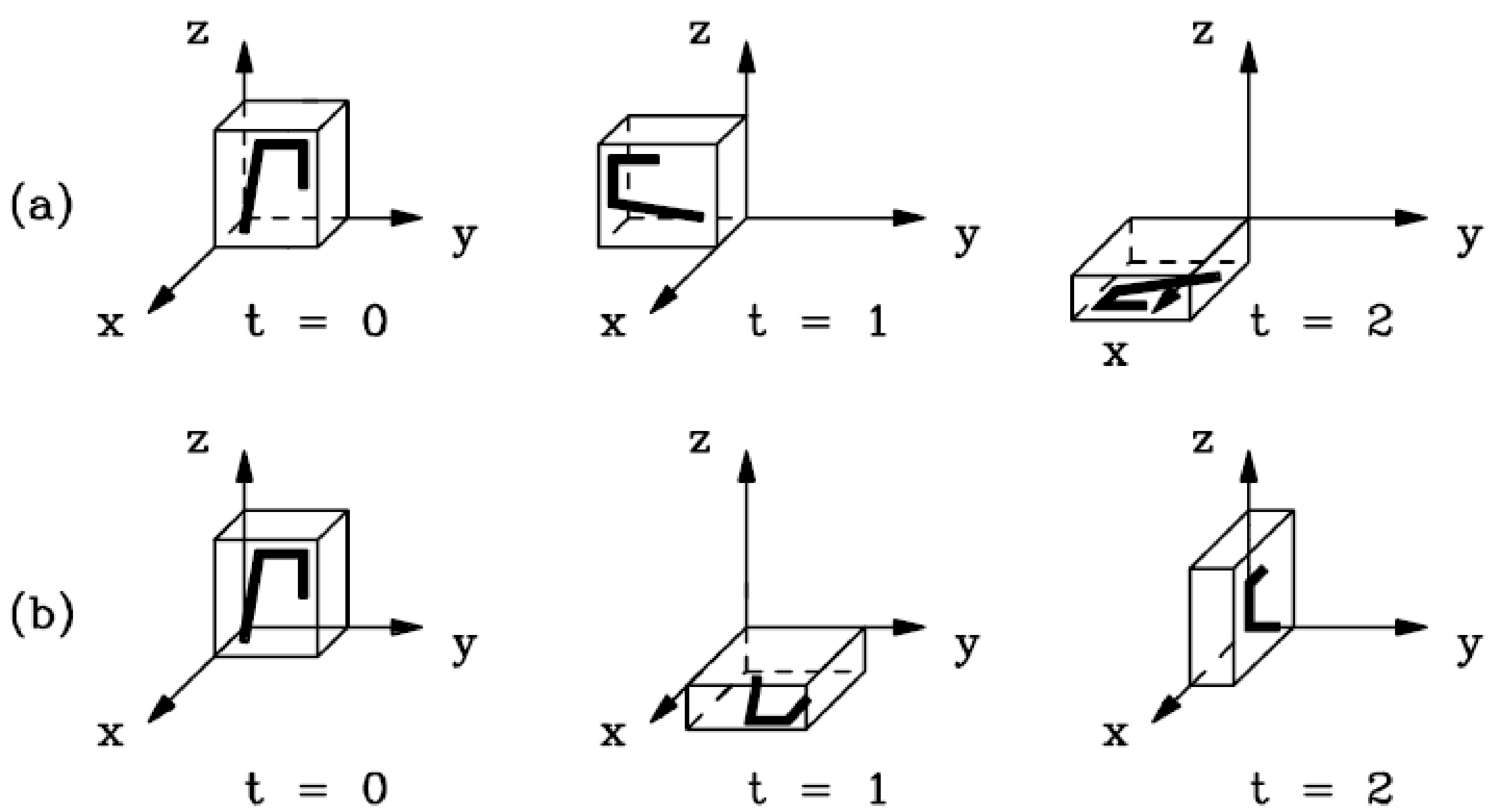
$$\begin{aligned} \boldsymbol{\omega} &= [\pi/2,\ 0,\ 0]^T & 0 \leq t \leq 1 \\ \boldsymbol{\omega} &= [0,\ \pi/2,\ 0]^T & 1 < t \leq 2 \end{aligned}$$

Case b)

$$\begin{aligned} \boldsymbol{\omega} &= [0,\ \pi/2,\ 0]^T & 0 \leq t \leq 1 \\ \boldsymbol{\omega} &= [\pi/2,\ 0,\ 0]^T & 1 < t \leq 2 \end{aligned}$$

By integrating the velocities in the two cases, one obtains:

$$\int_0^2 \boldsymbol{\omega} dt = [\pi/2,\ \pi/2,\ 0]^T$$



Case a)

$$\omega = [\pi/2,\ 0,\ 0]^T \qquad 0 \le t \le 1$$
$$\omega = [0,\ \pi/2,\ 0]^T \qquad 1 < t \le 2$$

Case b)

$$\omega = [0,\ \pi/2,\ 0]^T \qquad 0 \le t \le 1$$
$$\omega = [\pi/2,\ 0,\ 0]^T \qquad 1 < t \le 2$$

$$\int_0^2 \omega\, dt = [\pi/2,\ \pi/2,\ 0]^T$$

On the other hand, the rotation matrices in the two cases are:

$$R_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \qquad R_b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

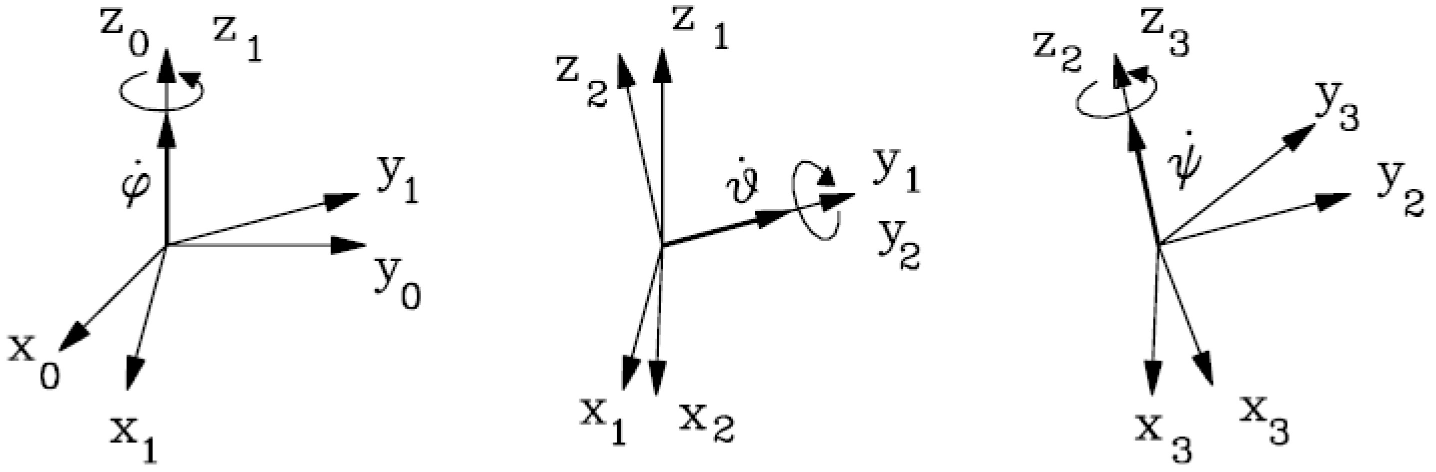
$\Rightarrow$ The integration of ω does not have a clear physical interpretation

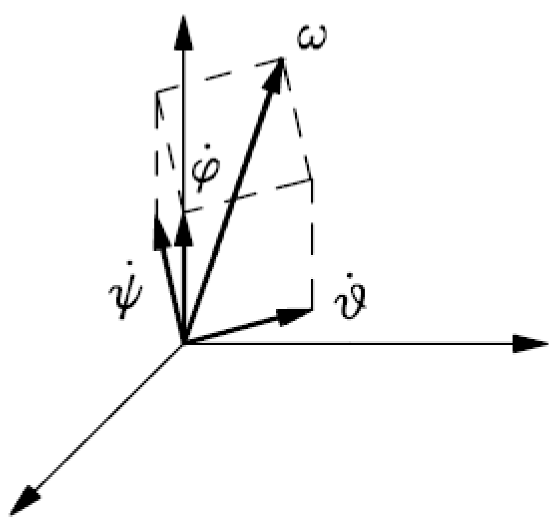
So γ is the winner? NO!

Problem 2: while $\boldsymbol{\omega}$ represents the velocity components about the three axes of $\mathcal{F}_0$, the elements of $\dot{\gamma}$ are defined with respect to a frame that:

a) is not Cartesian (its axes are not orthogonal to each other)

b) varies in time according to γ





Problem 2

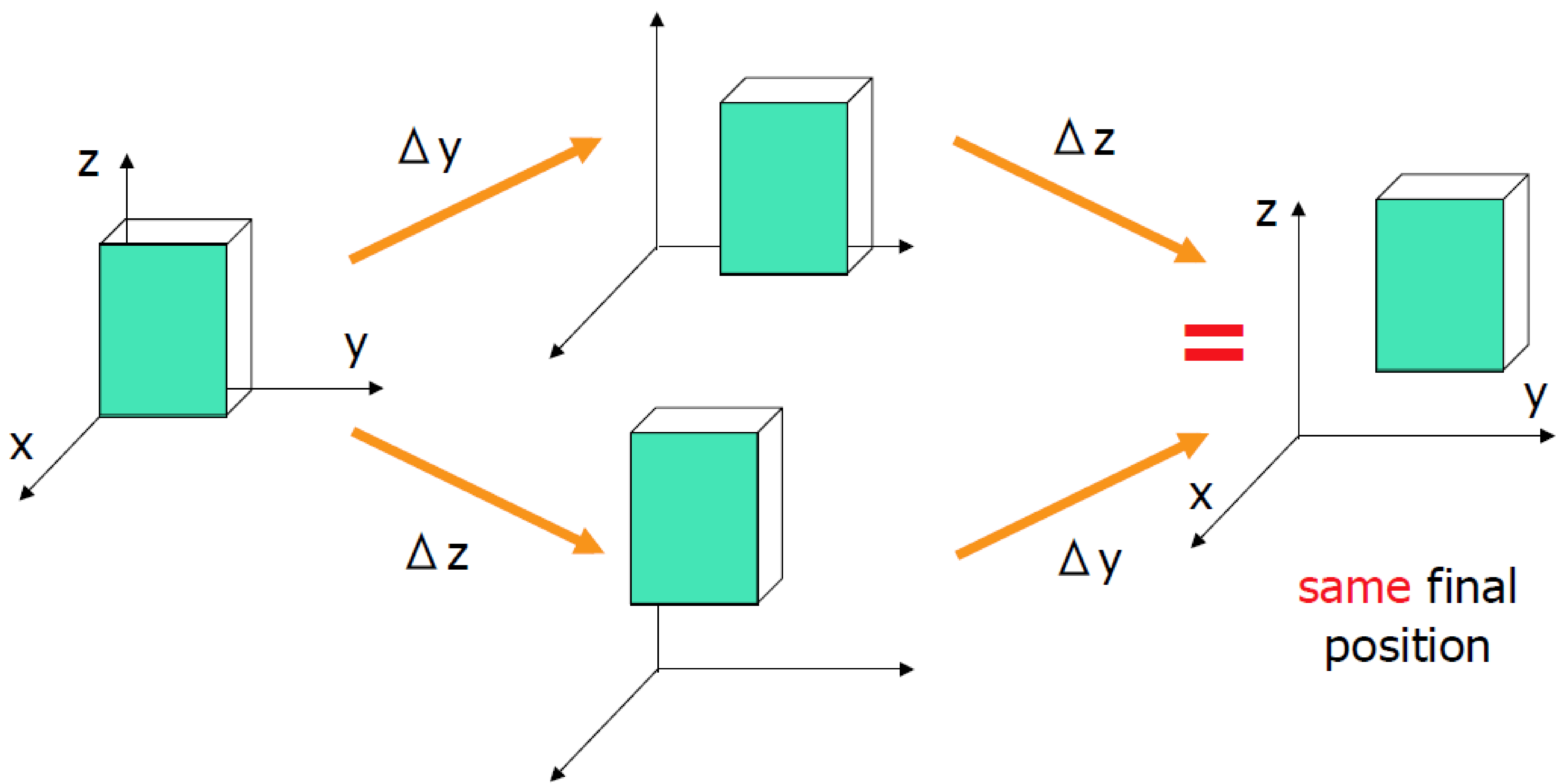
- v and ω are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - these contributions will be those of the joint velocities
- On the other hand, γ (and $d\gamma/dt$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

$\Rightarrow$ In general $\omega \neq d\gamma/dt$

However, the two expressions physically define the same phenomenon (velocity of the manipulator) and therefore a relationship between them must exist.

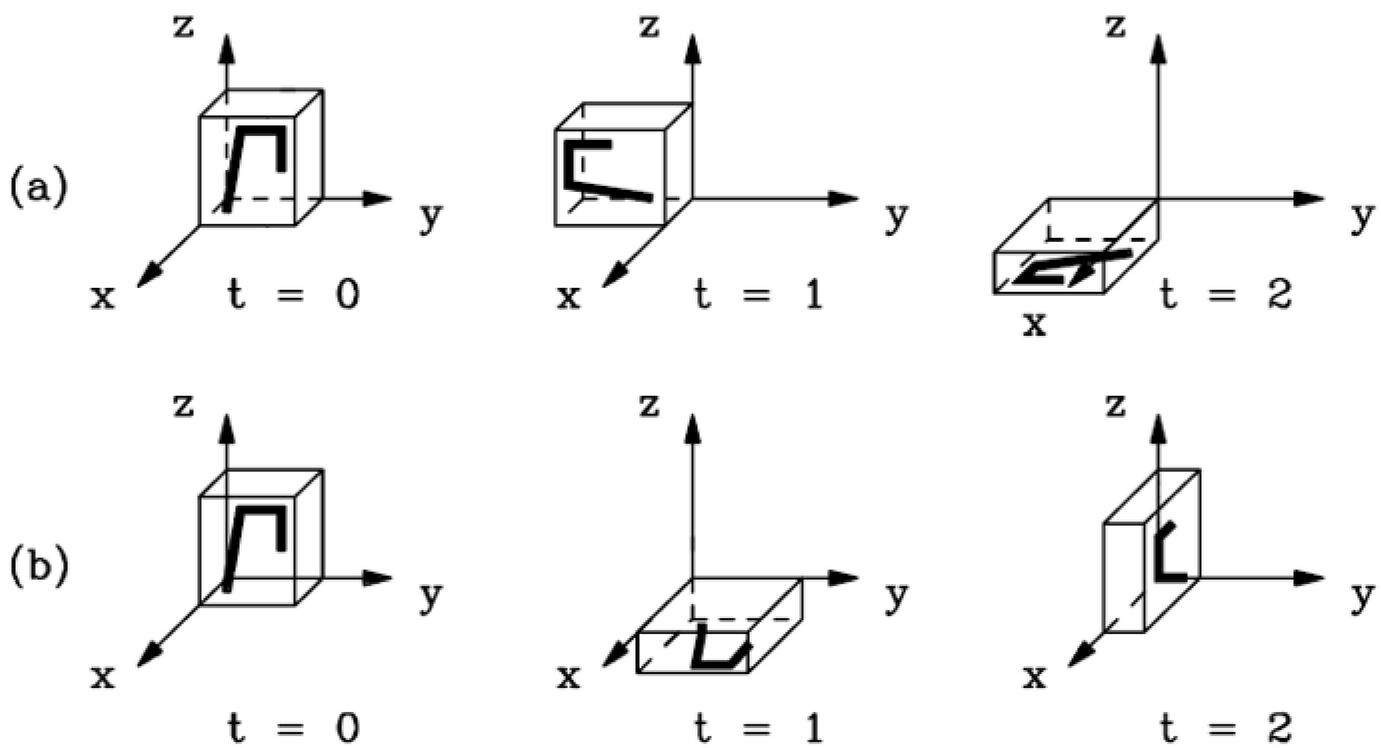
Finite and infinitesimal translations

Finite Δx, Δy, Δz or infinitesimal dx, dy, dz translations (linear displacements) always commute



Finite rotations do not commute

We just saw an example:



However…

Infinitesimal rotations do commute!

Infinitesimal rotations $d\varphi_X$, $d\varphi_Y$, $d\varphi_Z$ around x, y, z axes

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_Y(\phi_Y) = \begin{bmatrix} \cos\phi_Y & 0 & \sin\phi_Y \\ 0 & 1 & 0 \\ -\sin\phi_Y & 0 & \cos\phi_Y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_Z(\phi_Z) = \begin{bmatrix} \cos\phi_Z & -\sin\phi_Z & 0 \\ \sin\phi_Z & \cos\phi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

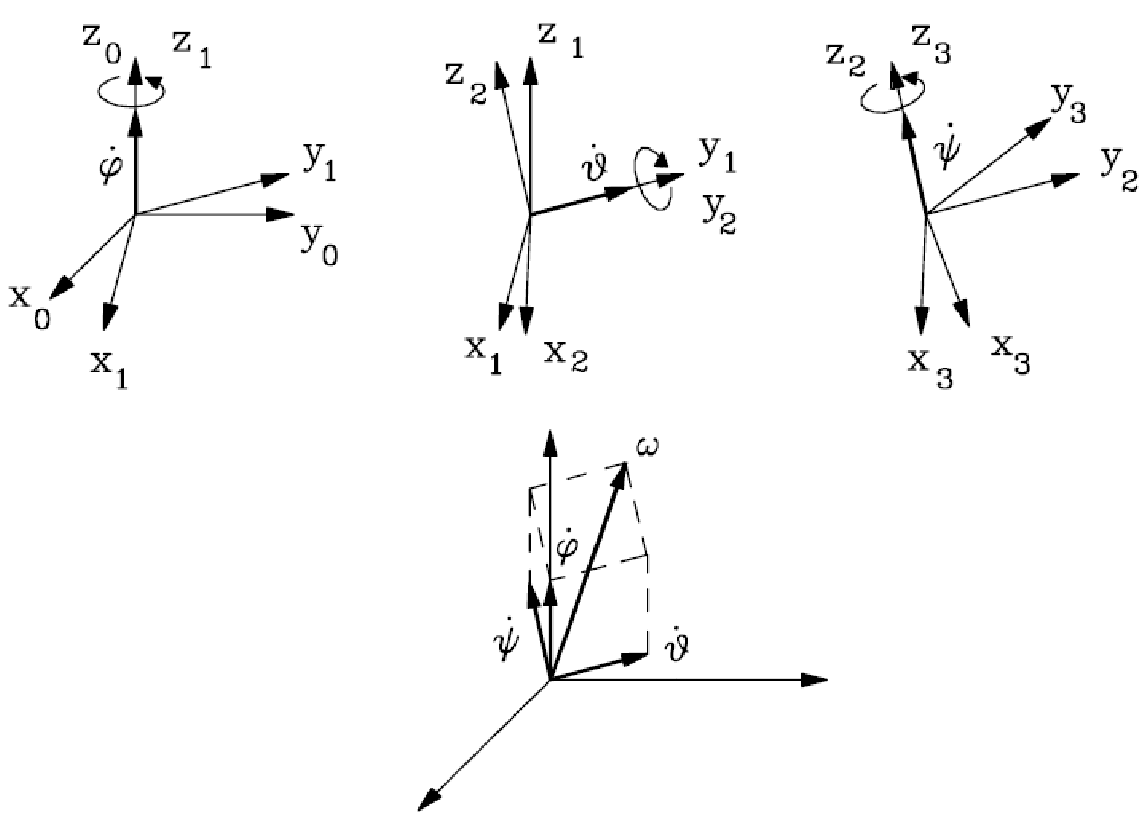
- $R(d\phi) = R(d\phi_X, d\phi_Y, d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix}$ ← neglecting second- and third-order (infinitesimal) terms

In summary

The two expressions of the Jacobian matrix physically define the same phenomenon (velocity of the manipulator) and therefore a relationship between them must exist

For example, if the Euler angles φ, θ, ψ are used for the triplet γ, it is possible to show that

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \dot{\gamma} \; = \; \mathbf{T}(\gamma)\, \dot{\gamma}$$



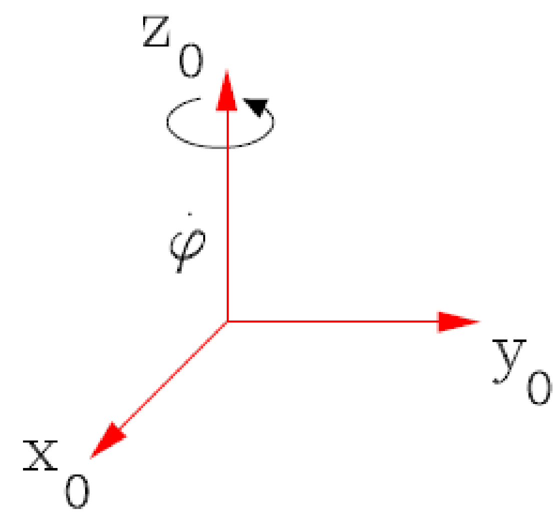
Note that matrix $T(\gamma)$ is singular when $\sin\theta = 0$. In this case, some rotational velocities may be expressed by ω and not by $\dot{\gamma}$, e.g.

$$\omega = [\cos\varphi, \sin\varphi, 0]^T$$

These cases are called **representation singularities** of $\boldsymbol{\gamma}$.

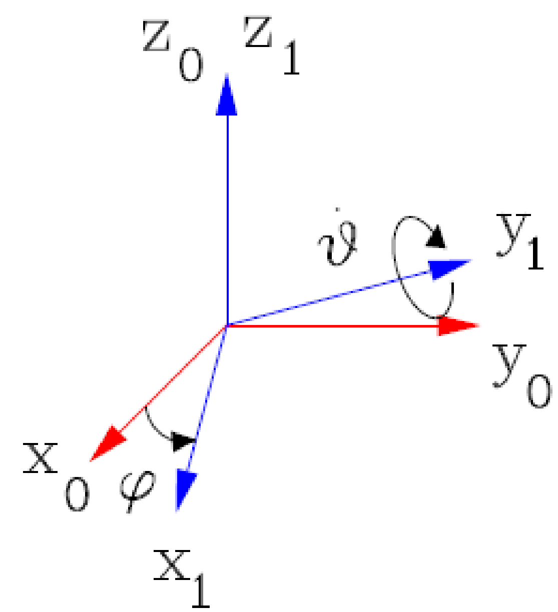
Definition of matrix T(γ):

$$\dot{\phi}: \quad [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\phi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



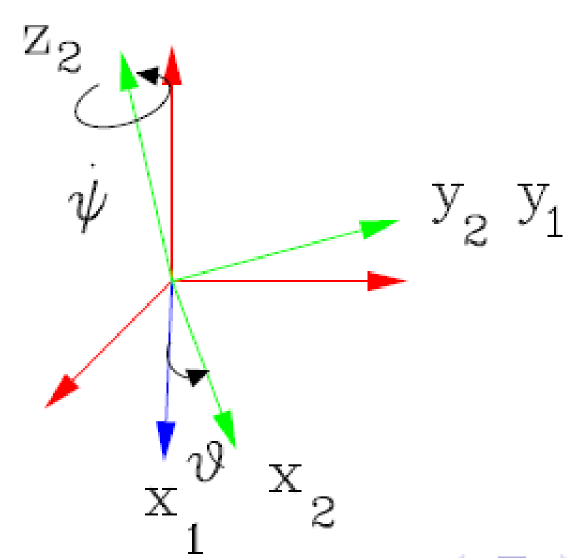
$$\omega_z = \dot{\phi}$$

$$\dot{\theta}: \quad [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\theta}\begin{bmatrix} -S_\phi \\ C_\phi \\ 0 \end{bmatrix}$$



$$\begin{cases} \omega_x = -S_\phi \dot{\theta} \\ \omega_y = C_\phi \dot{\theta} \end{cases}$$

$$\dot{\psi}: \quad [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\psi}\begin{bmatrix} -C_\phi S_\theta \\ S_\phi S_\theta \\ C_\theta \end{bmatrix}$$



$$\begin{cases} \omega_z = C_\theta \dot{\psi} \\ \omega_x = S_\theta C_\phi \dot{\psi} \\ \omega_y = S_\theta S_\phi \dot{\psi} \end{cases}$$

If $\sin\theta = 0$, then the components perpendicular to **z** of the velocity expressed by $\dot{\gamma}$ are linearly dependent $\left(\omega_x^2 + \omega_y^2 = \dot{\theta}^2\right)$, while physically this constraint may not exist!

From:

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \dot{\gamma}$$

one obtains:

$$\begin{bmatrix} 0 & -S_\phi & 0 \\ 0 & C_\phi & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \Longrightarrow \begin{bmatrix} -S_\phi\dot{\theta} \\ C_\phi\dot{\theta} \\ \dot{\phi}+\dot{\psi} \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \Longrightarrow \begin{cases} \omega_x^2 + \omega_y^2 = \dot{\theta}^2 \\ \omega_z = \dot{\phi} + \dot{\psi} \end{cases}$$

Finally...:

In general, given a triplet of angles γ, a transformation matrix $\mathbf{T}(\gamma)$ exists such that

$$\omega = \mathbf{T}(\gamma)\, \dot{\gamma}$$

Once the matrix $\mathbf{T}(\gamma)$ is known, it is possible to relate the analytical and geometrical expressions of the Jacobian matrix:

$$\begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\gamma) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\gamma} \end{bmatrix}$$

Then

$$\mathbf{J}_G = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\gamma) \end{bmatrix} \mathbf{J}_A = \mathbf{T}_A(\gamma)\mathbf{J}_A$$

Until now:

- We saw how we can define velocities in a robot/rigid-body environment

- We know the connection between the analytical Jacobian and the geometric Jacobian

$$\mathbf{J}_G = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\gamma) \end{bmatrix} \mathbf{J}_A = \mathbf{T}_A(\gamma)\mathbf{J}_A$$

- Now we calculate both of them

Analytical Jacobian

The analytical expression of the Jacobian is obtained by differentiating a vector $\mathbf{x} = \mathbf{f}(\mathbf{q}) \in \mathbb{R}^6$, that defines the position and orientation (according to some convention) of the manipulator in $\mathcal{F}_0$

By differentiating $\mathbf{f}(\mathbf{q})$, one obtains

$$
\begin{aligned}
d\mathbf{x} &= \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} d\mathbf{q} \\
&= \mathbf{J}(\mathbf{q}) d\mathbf{q}
\end{aligned}
$$

where the $m \times n$ matrix

$$
\mathbf{J}(\mathbf{q}) = \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \cdots & & & \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \qquad \mathbf{J}(\mathbf{q}) \in \mathbb{R}^{m \times n}
$$

is called the ***Jacobian matrix*** or ***JACOBIAN*** of the manipulator

Analytical Jacobian

If the infinitesimal period of time *dt* is considered, one obtains

$$\frac{d\,\mathbf{x}}{dt} = \mathbf{J}(\mathbf{q})\frac{d\,\mathbf{q}}{dt}$$

that is

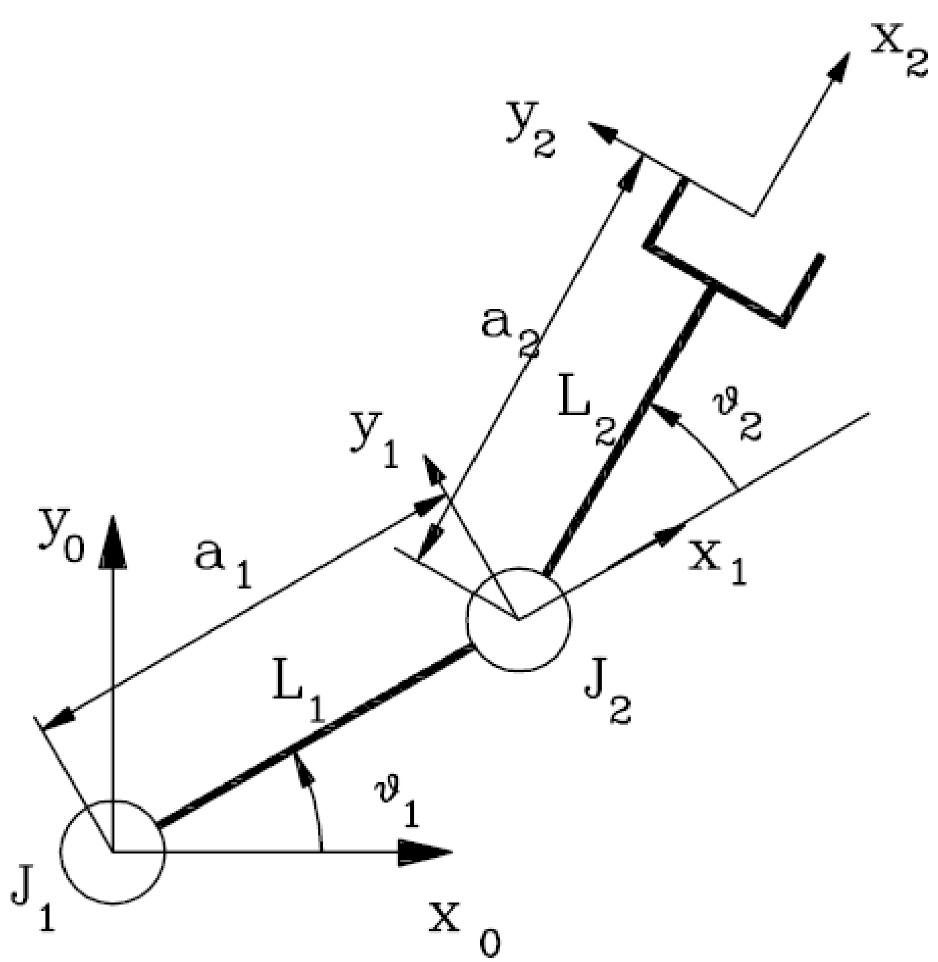
$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v} \\ \dot{\gamma} \end{bmatrix} = \mathbf{J}(\mathbf{q})\ \dot{\mathbf{q}}$$

relating the velocity vector $\dot{\mathbf{x}}$ expressed in $\mathcal{F}_0$ and the joint velocity vector $\dot{\mathbf{q}}$

- The elements $J_{i,j}$ of the Jacobian are nonlinear functions of $\mathbf{q}(t)$: therefore these elements change in time according to the value of the joint variables
- The Jacobian's dimensions depend on the number *n* of joints and on the dimension *m* of the considered operative space:

$$\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{m \times n}$$

AJ-Example: 2 DOF manipulator



	d	θ	a	α
L1	0	θ_1	a_1	0^o
L2	0	θ_2	a_2	0^o

The end-effector position is

$$\begin{aligned} p_x &= a_1 C_1 + a_2 C_{12} \\ p_y &= a_1 S_1 + a_2 S_{12} \\ p_z &= 0 \end{aligned}$$

If γ is composed by the Euler angles ϕ, θ, ψ defined about axes $\mathbf{z}_0, \mathbf{y}_1, \mathbf{z}_2$, and considering that the $\mathbf{z}$ axes of the base frame and of the end effector are parallel, the total rotation is equivalent to a single rotation about $\mathbf{z}_0$ and therefore:

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \theta_1 + \theta_2 \\ 0 \\ 0 \end{bmatrix}$$

AJ-Example: 2 DOF manipulator

Euler angles:

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \theta_1 + \theta_2 \\ 0 \\ 0 \end{bmatrix}$$

By differentiation of the expressions of $\mathbf{p}$ and γ one obtains

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{\mathbf{q}}$$

$$= \mathbf{J}(\mathbf{q})\, \dot{\mathbf{q}}$$

Geometric Jacobian

Geometric Expression of the Jacobian

- The geometric expression of the Jacobian is obtained considering the rotational velocity vector $\boldsymbol{\omega}$
- Each column of the Jacobian matrix defines the effect of the *i*-th joint on the end-effector velocity and it is divided in two terms
- The first term considers the effect of $\dot{q}_i$ on the the linear velocity $\mathbf{v}$, while the second one on the rotational velocity $\boldsymbol{\omega}$, i.e.

$$\begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} = \mathbf{J}\,\dot{\mathbf{q}} \qquad \Longrightarrow \qquad \mathbf{J} = \begin{bmatrix} \mathbf{J}_{v1} & \mathbf{J}_{v2} & \dots & \mathbf{J}_{vn} \\ \mathbf{J}_{\omega 1} & \mathbf{J}_{\omega 2} & \dots & \mathbf{J}_{\omega n} \end{bmatrix}$$

- Therefore

$$\begin{aligned} \mathbf{v} &= \mathbf{J}_{v1}\dot{q}_1 + \mathbf{J}_{v2}\dot{q}_2 + \ldots + \mathbf{J}_{vn}\dot{q}_n \\ \omega &= \mathbf{J}_{\omega 1}\dot{q}_1 + \mathbf{J}_{\omega 2}\dot{q}_2 + \ldots + \mathbf{J}_{\omega n}\dot{q}_n \end{aligned}$$

- ➢ The analytic and geometric Jacobian differ for the rotational part
- ➢ In order to obtain the geometric Jacobian, a general method based on the geometrical structure of the manipulator is adopted

Derivative of a Rotation Matrix

- Let's consider a rotation matrix $\mathbf{R} = \mathbf{R}(t)$ and $\mathbf{R}(t)\mathbf{R}^T(t) = \mathbf{I}$
- Let's derive the equation: $\mathbf{R}(t)\mathbf{R}^T(t) = \mathbf{I} \Rightarrow \dot{\mathbf{R}}(t)\mathbf{R}^T(t) + \mathbf{R}(t)\dot{\mathbf{R}}^T(t) = \mathbf{0}$
- A 3 × 3 (skew-symmetric) matrix $\mathbf{S}(t)$ is obtained

$$\mathbf{S}(t) = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)$$

- As a matter of fact

$$\mathbf{S}(t) + \mathbf{S}^T(t) = \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Then

$$\dot{\mathbf{R}}(t) = \mathbf{S}(t)\,\mathbf{R}(t)$$

- This means that the derivative of a rotation matrix is expressed as a function of the matrix itself

Derivative of a Rotation Matrix

Physical interpretation:
Matrix $\mathbf{S}(t)$ is expressed as a function of a vector $\boldsymbol{\omega}(t) = [\omega_x, \omega_y, \omega_z]^T$ representing the angular velocity of $\mathbf{R}(t)$

Consider a constant vector **p'** and the vector $\mathbf{p}(t) = \mathbf{R}(t)\mathbf{p}'$

The time derivative of $\mathbf{p}(t)$ is

$$\dot{\mathbf{p}}(t) = \dot{\mathbf{R}}(t)\mathbf{p}'$$

which can be written as

$$\dot{\mathbf{p}}(t) = \mathbf{S}(t)\mathbf{R}(t)\mathbf{p}' = \boldsymbol{\omega} \times (\mathbf{R}(t)\ \mathbf{p}')$$

(This last result is well known from the classical mechanics of rigid bodies)

Derivative of a Rotation Matrix

- Moreover it can be shown that:

$$\mathbf{R}\,\mathbf{S}(\omega)\,\mathbf{R}^T = \mathbf{S}(\mathbf{R}\,\omega)$$

i.e. the matrix form of $\mathbf{S}(\boldsymbol{\omega})$ in a frame rotated by $\mathbf{R}$ is the same as the skew-symmetric matrix $\mathbf{S}(\mathbf{R}\,\boldsymbol{\omega})$ of the vector $\boldsymbol{\omega}$ rotated by $\mathbf{R}$

- (1) Note also that $\mathbf{S}(\boldsymbol{\omega})$ is linear in its argument:

$$\mathbf{S}(k_1\boldsymbol{\omega}_1 + k_2\boldsymbol{\omega}_2) = k_1\mathbf{S}(\boldsymbol{\omega}_1) + k_2\mathbf{S}(\boldsymbol{\omega}_2)$$

- (2) Note also the property of $\mathbf{S}(\boldsymbol{\omega})$:

$$\mathbf{S}(\boldsymbol{\omega})\,\mathbf{p} = \boldsymbol{\omega} \times \mathbf{p}$$

Derivative of a Rotation Matrix

Consider two frames $\mathcal{F}$ and $\mathcal{F}'$, which differ by the rotation $\mathbf{R}$ ($\boldsymbol{\omega}' = \mathbf{R}\,\boldsymbol{\omega}$)

Then $\mathbf{S}(\boldsymbol{\omega}')$ operates on vectors in $\mathcal{F}'$ and $\mathbf{S}(\boldsymbol{\omega})$ on vectors in $\mathcal{F}$

Consider a vector $\mathbf{v}_a{}'$ in $\mathcal{F}'$ and assume that some operations must be performed on that vector in $\mathcal{F}$ (then using $\mathbf{S}$)

It is necessary to:

1. Transform the vector(s) from $\mathcal{F}'$ to $\mathcal{F}$ (by $\mathbf{R}^T$)
2. Use $\mathbf{S}(\boldsymbol{\omega})$
3. Transform back the result to $\mathcal{F}'$ (by $\mathbf{R}$)

That is:

$$\mathbf{v}'_b = \mathbf{R}\ \mathbf{S}(\omega)\ \mathbf{R}^T\ \mathbf{v}'_a$$
$$\mathbf{v}'_b = \mathbf{S}(\omega')\ \mathbf{v}'_a$$

equivalent to the transformation using $\mathbf{S}(\boldsymbol{\omega})$

Example

Consider the elementary rotation about $\mathbf{z}$

$$Rot(\mathbf{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If θ is a function of time

$$\mathbf{S}(t) = \begin{bmatrix} -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta & 0 \\ \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{S}(\omega(t))$$

Then

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

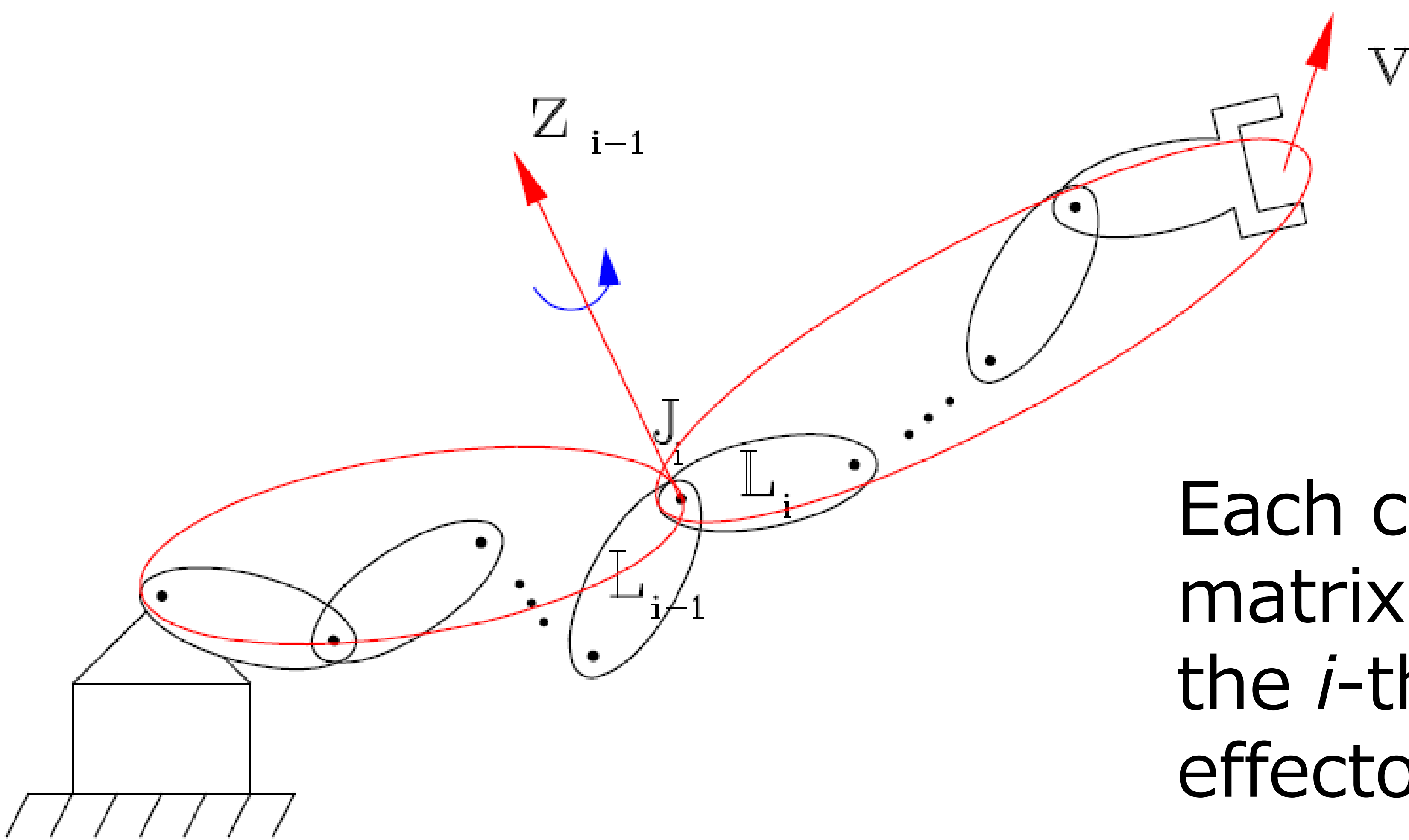
i.e. a rotational velocity about $\mathbf{z}$.

Geometric Jacobian

The end-effector velocity is a linear composition of the joint velocities

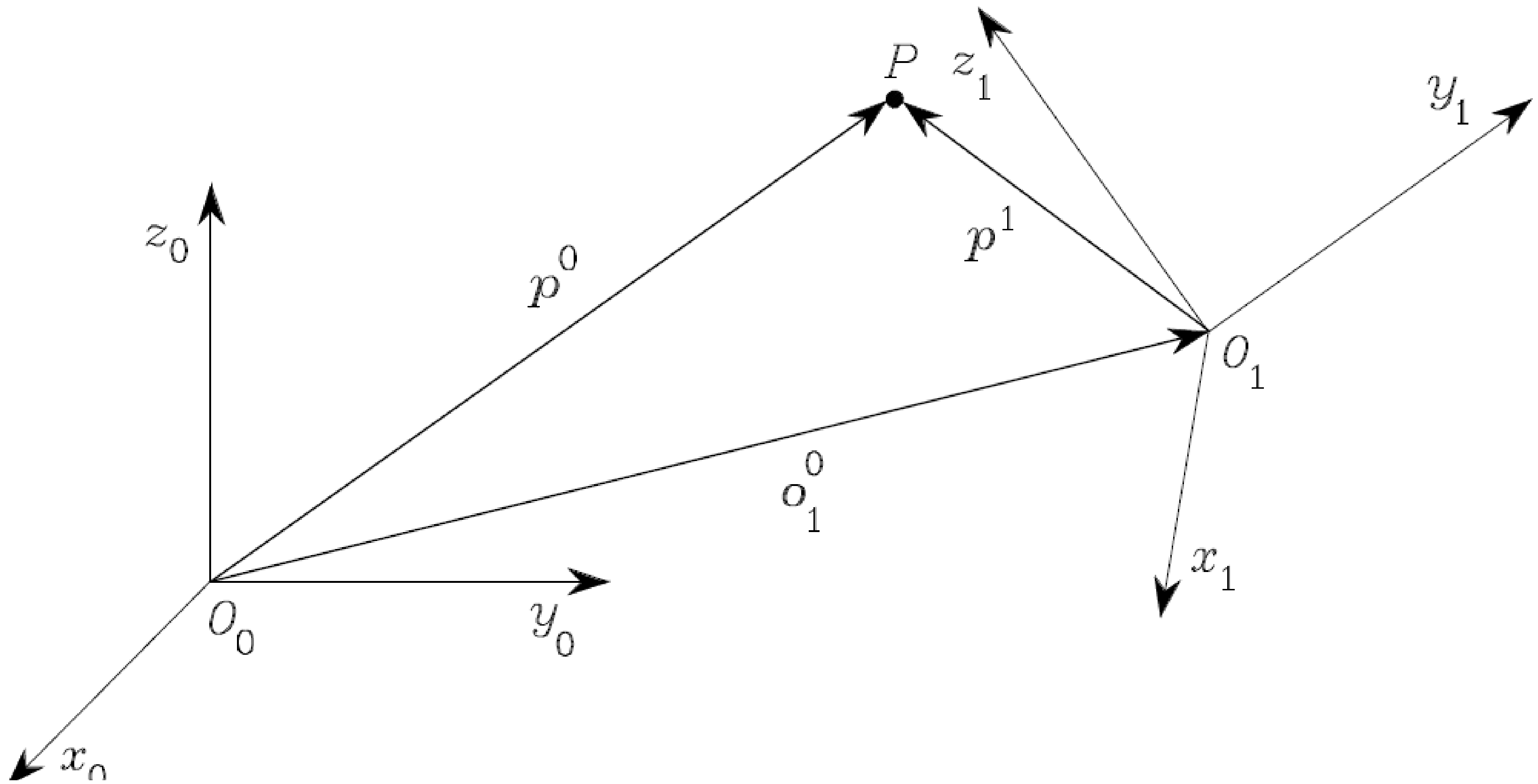
$$\mathbf{v} = \mathbf{J}_{v1}\dot{q}_1 + \mathbf{J}_{v2}\dot{q}_2 + \ldots + \mathbf{J}_{vn}\dot{q}_n$$

$$\omega = \mathbf{J}_{\omega 1}\dot{q}_1 + \mathbf{J}_{\omega 2}\dot{q}_2 + \ldots + \mathbf{J}_{\omega n}\dot{q}_n$$



Each column of the Jacobian matrix defines the effect of the *i*-th joint on the end-effector velocity

Geometric Jacobian



$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\begin{aligned} \dot{\boldsymbol{p}}^0 &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{R}}_1^0 \boldsymbol{p}^1 \\ &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{S}(\boldsymbol{\omega}_1^0) \boldsymbol{R}_1^0 \boldsymbol{p}^1 \\ &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0 \end{aligned}$$

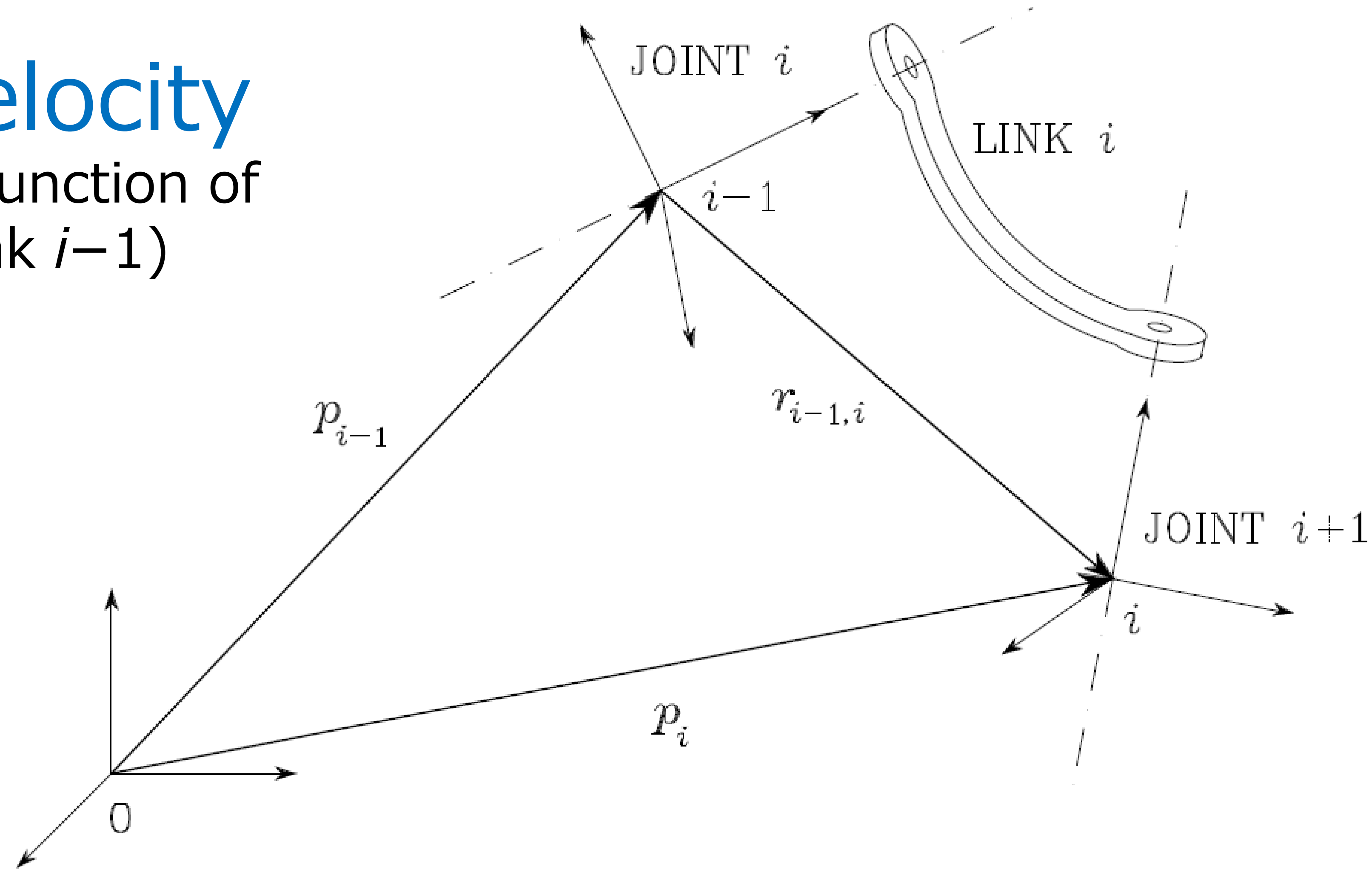
$$(\boldsymbol{r}_1^0 \triangleq \boldsymbol{R}_1^0 \boldsymbol{p}^1)$$

$$\boxed{\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0}$$

Geometric Jacobian – Link Velocity

Linear velocity

(of link *i* as a function of velocities of link *i*−1)



$\boldsymbol{v}_{i-1,i}$ denotes the velocity of the origin of Frame *i* with respect to the origin of Frame *i*−1

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}$$

$$\dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0\dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0$$

$$\begin{aligned}\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1} \\ &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}\end{aligned}$$

Geometric Jacobian – Link Velocity

Angular velocity

(of link *i* as a function of velocities of link *i*−1)

$$\boldsymbol{R}_i = \boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1}$$

$$\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i = \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1}$$

$$= \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{S}(\boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1}$$

$$= \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$

Geometric Jacobian – Link Velocity

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$
$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}$$

Prismatic joint:

$$\boldsymbol{\omega}_{i-1,i} = \mathbf{0}$$
$$\boldsymbol{v}_{i-1,i} = \dot{d}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1}$$
$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

Revolute joint:

$$\boldsymbol{\omega}_{i-1,i} = \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$
$$\boldsymbol{v}_{i-1,i} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,i}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$
$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

Geometric Jacobian – Computation

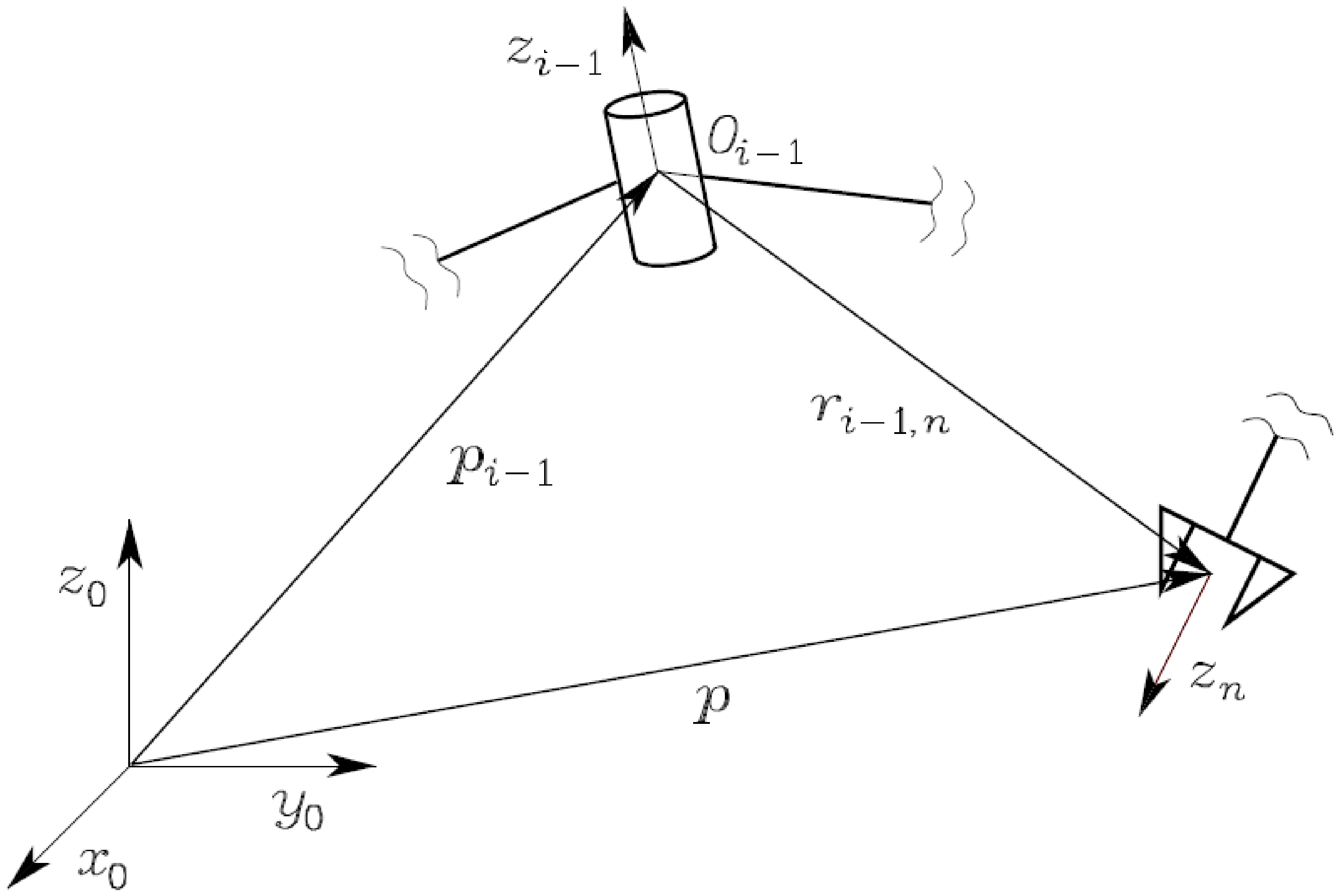
$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{\jmath}_{P1} & & \boldsymbol{\jmath}_{Pn} \\ & \dots & \\ \boldsymbol{\jmath}_{O1} & & \boldsymbol{\jmath}_{On} \end{bmatrix} \qquad \dot{v} = \sum_{i=1}^{n} \frac{\partial v}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} J_{P_i} \dot{q}_i$$

Linear velocity

Joint *i* ***prismatic***

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \dot{d}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1}$$

Joint *i* ***revolute***



$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,n}$$
$$= \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1})$$
$$\Downarrow$$
$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1})$$

Geometric Jacobian – Computation

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{\jmath}_{P1} & & \boldsymbol{\jmath}_{Pn} \\ & \dots & \\ \boldsymbol{\jmath}_{O1} & & \boldsymbol{\jmath}_{On} \end{bmatrix} \qquad \boldsymbol{\omega}_e = \boldsymbol{\omega}_n = \sum_{i=1}^{n} \boldsymbol{\omega}_{i-1,i} = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Oi}\dot{q}_i$$

Angular velocity

Joint *i* ***prismatic***

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \mathbf{0} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Oi} = \mathbf{0}$$

Joint *i* ***revolute***

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Oi} = \boldsymbol{z}_{i-1}$$

Geometric Jacobian – Computation

Column of geometric Jacobian

$$\begin{bmatrix} \boldsymbol{j}_{Pi} \\ \boldsymbol{j}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix} & \textit{prismatic}\ \text{joint} \\ \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix} & \textit{revolute}\ \text{joint} \end{cases}$$

$$\star\ \boldsymbol{z}_{i-1} = \boldsymbol{R}_1^0(q_1) \dots \boldsymbol{R}_{i-1}^{i-2}(q_{i-1}) \boldsymbol{z}_0$$

$$\star\ \tilde{\boldsymbol{p}} = \boldsymbol{A}_1^0(q_1) \dots \boldsymbol{A}_n^{n-1}(q_n) \tilde{\boldsymbol{p}}_0$$

$$\star\ \tilde{\boldsymbol{p}}_{i-1} = \boldsymbol{A}_1^0(q_1) \dots \boldsymbol{A}_{i-1}^{i-2}(q_{i-1}) \tilde{\boldsymbol{p}}_0$$

Geometric Jacobian – Representation in a Different Frame

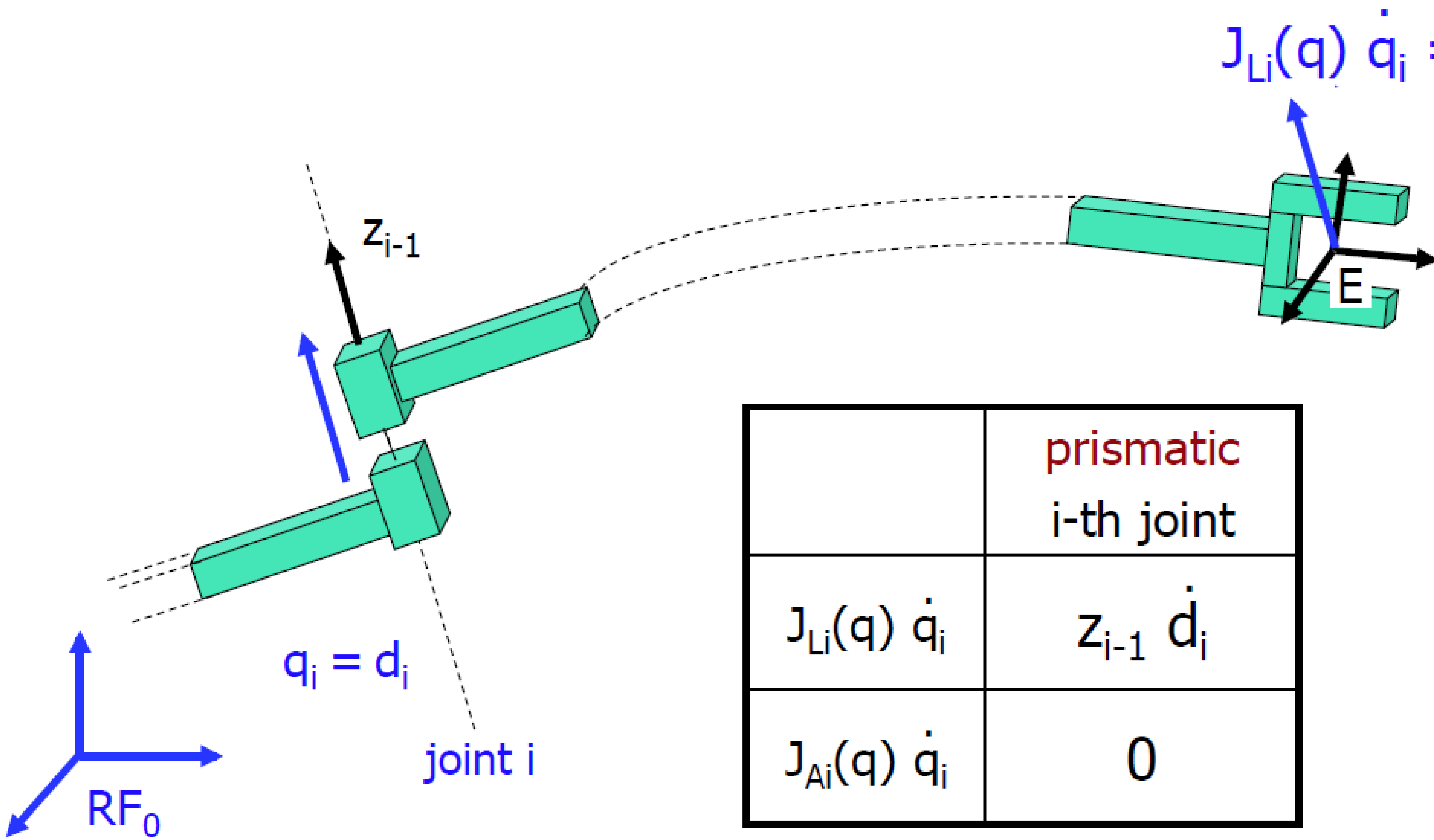
- The Jacobian matrix depends on the frame in which the end-effector velocity is expressed
- The above equations allow computation of the geometric Jacobian with respect to the base frame
- For a different Frame t:

$$\begin{bmatrix} \dot{\boldsymbol{p}}^t \\ \boldsymbol{\omega}^t \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} \\ \boldsymbol{\omega} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \boldsymbol{J}\dot{\boldsymbol{q}}$$

$$\boldsymbol{J}^t = \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \boldsymbol{J}$$

RECAP: Geometric Jacobian – Contribution of a Prismatic Joint

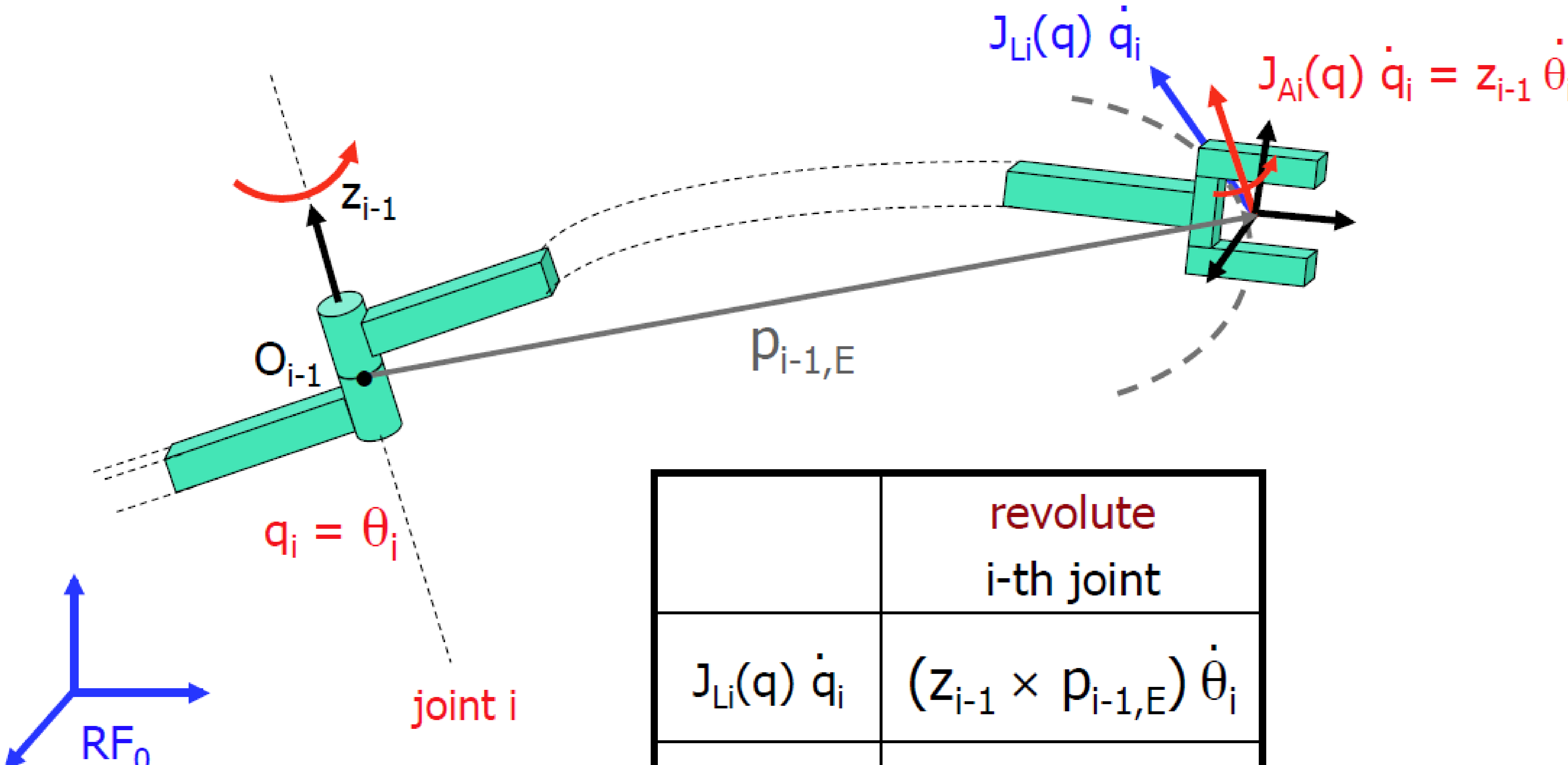
Note: joints beyond the *i*-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body



	prismatic i-th joint
$J_{Li}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{d}_i$
$J_{Ai}(q)\,\dot{q}_i$	0

RECAP: Geometric Jacobian – Contribution of a Revolute Joint

Note: joints beyond the *i*-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body



	revolute i-th joint
$J_{Li}(q)\,\dot{q}_i$	$(z_{i-1} \times p_{i-1,E})\,\dot{\theta}_i$
$J_{Ai}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{\theta}_i$

RECAP: Geometric Jacobian

It is possible to show that the i-th column of the Jacobian can be computed as

$$\begin{bmatrix} \mathbf{J}_{vi} \\ \mathbf{J}_{\omega i} \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{z}_{i-1} \times \left({}^0\mathbf{p}_n - {}^0\mathbf{p}_{i-1}\right) \\ {}^0\mathbf{z}_{i-1} \end{bmatrix} \quad \text{revolute joint}$$

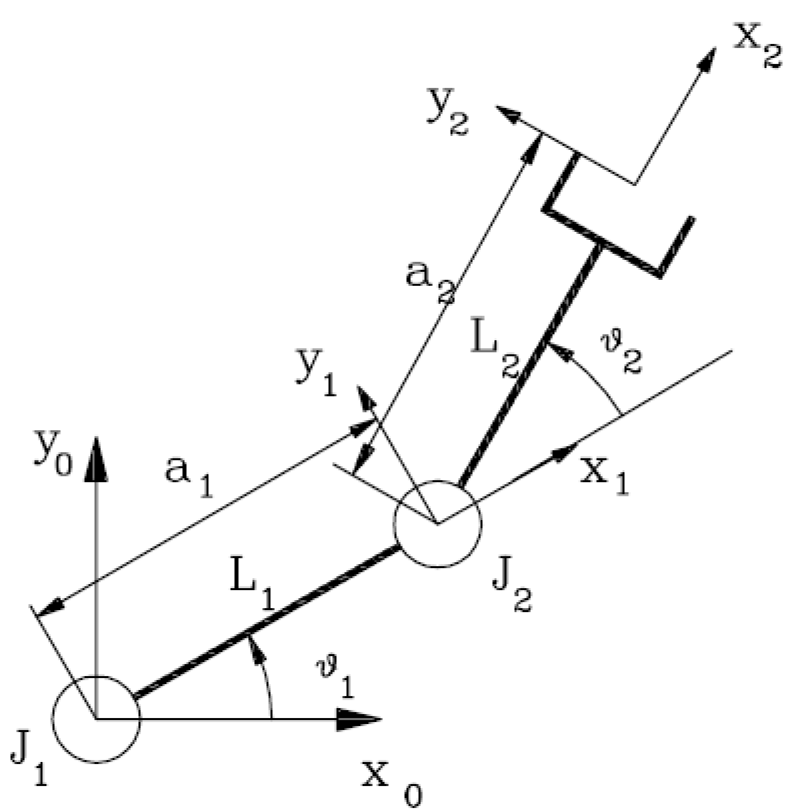
$$\begin{bmatrix} \mathbf{J}_{vi} \\ \mathbf{J}_{\omega i} \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{z}_{i-1} \\ \mathbf{0} \end{bmatrix} \quad \text{prismatic joint}$$

where ${}^0\mathbf{z}_{i-1}$ and ${}^0\mathbf{r}_{i-1,n} = {}^0\mathbf{p}_n - {}^0\mathbf{p}_{i-1}$ depend on the joint variables $q_1, q_2, ..., q_n$. In particular:

- ${}^0\mathbf{p}_n$ is the end-effector position, defined in the first three elements of the last column of ${}^0\mathbf{T}_n = {}^0\mathbf{H}_1(q_1)\dots{}^{n-1}\mathbf{H}_n(q_n)$;
- ${}^0\mathbf{p}_{i-1}$ is the position of $\mathcal{F}_{i-1}$, defined in the first three elements of the last column of ${}^0\mathbf{T}_{i-1} = {}^0\mathbf{H}_1(q_1)\dots{}^{i-2}\mathbf{H}_{i-1}(q_{i-1})$;
- ${}^0\mathbf{z}_{i-1}$ is the third column of ${}^0\mathbf{R}_{i-1}$:

$${}^0\mathbf{R}_{i-1} = {}^0\mathbf{R}_1(q_1)\,{}^1\mathbf{R}_2(q_2)\dots{}^{i-2}\mathbf{R}_{i-1}(q_{i-1})$$

GJ-Example: 2 DOF manipulator



The Jacobian is computed as

$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_2 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) \\ \mathbf{z}_0 & \mathbf{z}_1 \end{bmatrix}$$

The origins of the frames are

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{p}_1 = \begin{bmatrix} a_1 C_1 \\ a_1 S_1 \\ 0 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ 0 \end{bmatrix}$$

and the rotational axes are

$$\mathbf{z}_0 = \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

GJ-Example: 2 DOF manipulator

Then

$$\mathbf{z}_0 \times (\mathbf{p}_2 - \mathbf{p}_0) = -\begin{bmatrix} 0 & 0 & a_1 S_1 + a_2 S_{12} \\ 0 & 0 & -a_1 C_1 - a_2 C_{12} \\ -a_1 S_1 - a_2 S_{12} & a_1 C_1 + a_2 C_{12} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 S_1 - a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} \\ 0 \end{bmatrix}$$

$$\mathbf{z}_1 \times (\mathbf{p}_2 - \mathbf{p}_1) = -\begin{bmatrix} 0 & 0 & a_2 S_{12} \\ 0 & 0 & -a_2 C_{12} \\ -a_2 S_{12} & a_2 C_{12} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a_2 S_{12} \\ a_2 C_{12} \\ 0 \end{bmatrix}$$

In conclusion:

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

GJ-Example: 2 DOF manipulator

Jacobian:

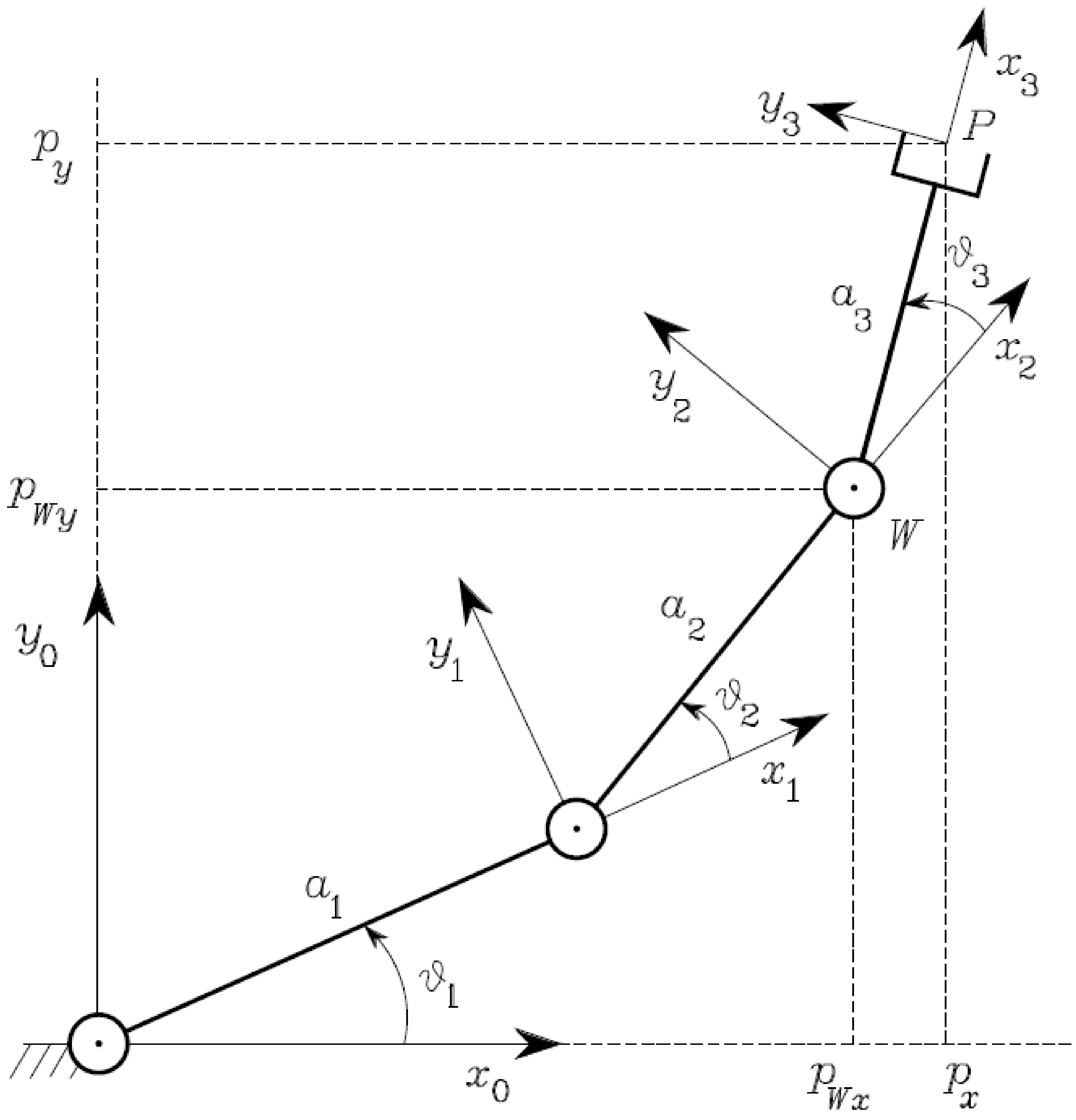
$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

If the orientation is not of interest, only the first two rows may be considered

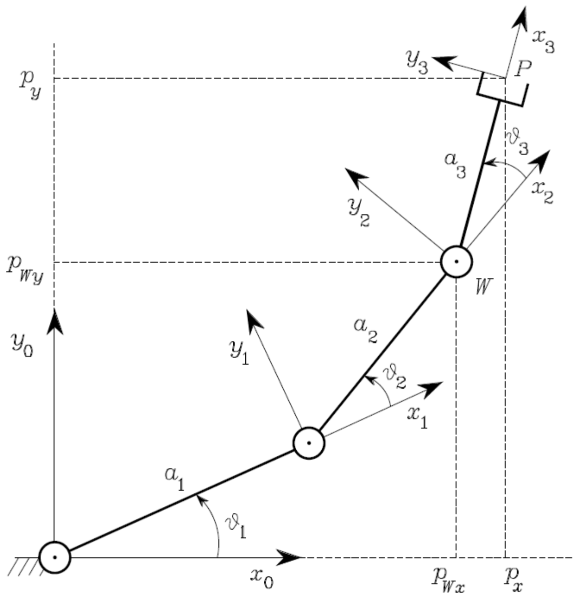
$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} \end{bmatrix}$$

maximum rank is 2 $\Rightarrow$ at most **2** components of the linear/angular end-effector velocity can be independently assigned

GJ-Example: 3-link planar manipulator



GJ-Example: 3-link planar manipulator



$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}$$

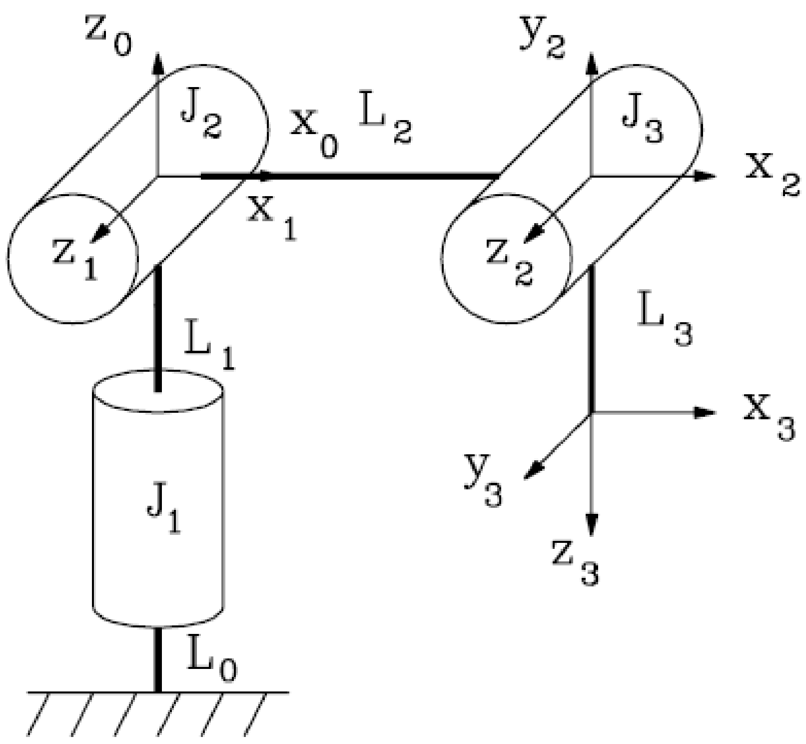
$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

GJ-Example: 3-link planar manipulator

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{J}_P = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \end{bmatrix}$$

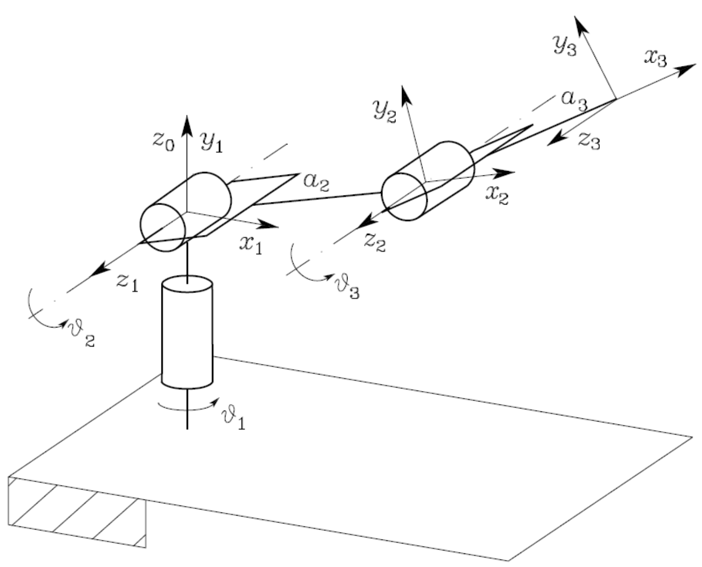
GJ-Example: 3 DOF anthropomorphic manipulator



The canonical transformation matrices are

$$
{}^{0}\mathbf{H}_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}\mathbf{H}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}\mathbf{H}_3 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3 C_3 \\ S_3 & C_3 & 0 & a_3 S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



and the kinematic model

$$
{}^{0}\mathbf{T}_3 = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & C_1(a_2 C_2 + a_3 C_{23}) \\ S_1 C_{23} & -S_1 S_{23} & -C_1 & S_1(a_2 C_2 + a_3 C_{23}) \\ S_{23} & C_{23} & 0 & a_2 S_2 + a_3 S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

GJ-Example: 3 DOF anthropomorphic manipulator

The Jacobian results

$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_3 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_3 - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_3 - \mathbf{p}_2) \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 \end{bmatrix}$$

where

$$\mathbf{p}_0 = \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} a_2 C_1 C_2 \\ a_2 S_1 S_2 \\ a_2 S_2 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} C_1(a_2 C_2 + a_3 C_{23}) \\ S_1(a_2 C_2 + a_3 C_{23}) \\ a_2 S_2 + a_3 S_{23} \end{bmatrix}$$

The rotational axes are

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix}$$

GJ-Example: 3 DOF anthropomorphic manipulator

Therefore

$$\mathbf{J} = \begin{bmatrix} -S_1(a_2C_2 + a_3C_{23}) & -C_1(a_2S_2 + a_3S_{23}) & -a_3C_1S_{23} \\ C_1(a_2C_2 + a_3C_{23}) & -S_1(a_2S_2 + a_3S_{23}) & -a_3S_1S_{23} \\ 0 & a_2C_2 + a_3C_{23} & a_3C_{23} \\ 0 & S_1 & S_1 \\ 0 & -C_1 & -C_1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Only three rows are linearly independent (3 dof).
- Note that it is not possible to achieve all the rotational velocities ω in $\mathbb{R}^3$.
- Moreover $S_1\omega_y = -C_1\omega_x \quad (\omega_x = S_1\dot{\theta}_2 + S_1\dot{\theta}_3,\ \omega_y = -C_1\dot{\theta}_2 - C_1\dot{\theta}_3,\ )$.

By considering the linear velocity only, one obtains:

$$\mathbf{J} = \begin{bmatrix} -S_1(a_2C_2 + a_3C_{23}) & -C_1(a_2S_2 + a_3S_{23}) & -a_3C_1S_{23} \\ C_1(a_2C_2 + a_3C_{23}) & -S_1(a_2S_2 + a_3S_{23}) & -a_3S_1S_{23} \\ 0 & a_2C_2 + a_3C_{23} & a_3C_{23} \end{bmatrix}$$

GJ-Example: 3 DOF anthropomorphic manipulator

Note that:

- $\dot{\theta}_1$ does not affect v_z (nor $\omega_x, \ \omega_y$)
- ω_z depends only by $\dot{\theta}_1$
- $S_1\omega_y = -C_1\omega_x$: $\quad$ ω_x and ω_y are not independent
- the first three rows may also be obtained by derivation of ${}^0\mathbf{p}_3$

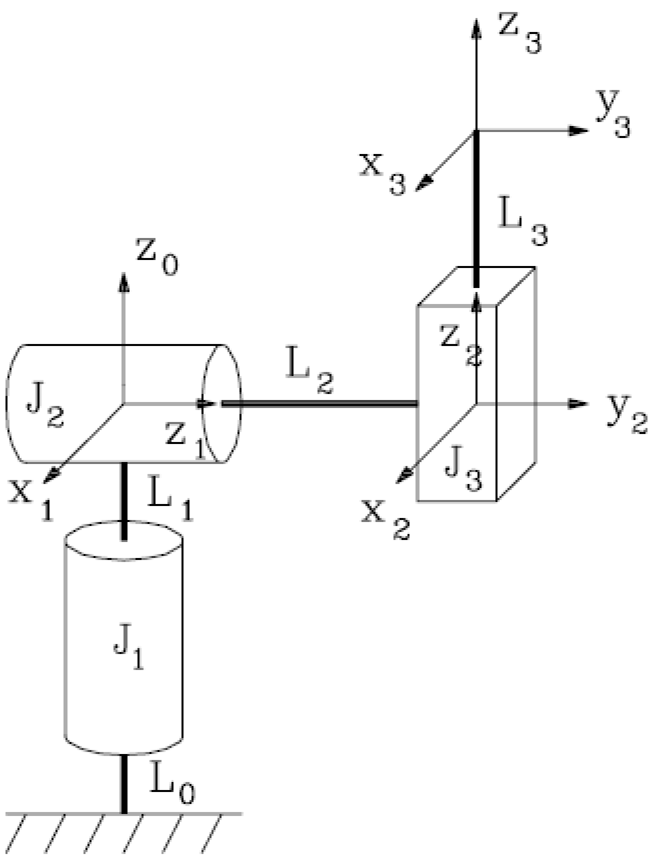
In the "linear velocity" case (i.e. the first three rows only)

$$det(\mathbf{J}) = -a_2 a_3 S_3 (a_2 C_2 + a_3 C_{23})$$

Therefore $det(\mathbf{J}) = 0$ in two cases:

- $S_3 = 0 \quad \Longrightarrow \quad \theta_3 = \begin{cases} 0 \\ \pi \end{cases}$
- $(a_2 C_2 + a_3 C_{23}) = 0$ i.e. when the wrist is on the z axis ($p_x = p_y = 0$): shoulder singularity

GJ-Example: 3 DOF spherical manipulator



Canonical transformation matrices

$$
{}^{0}\mathbf{H}_1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{1}\mathbf{H}_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}\mathbf{H}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Kinematic model:

$$
{}^{0}\mathbf{T}_3 = \begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 & -d_2 S_1 + d_3 C_1 S_2 \\ C_2 S_1 & C_1 & S_1 S_2 & d_2 C_1 + d_3 S_1 S_2 \\ -S_2 & 0 & C_2 & C_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

GJ-Example: 3 DOF spherical manipulator

The Jacobian is

$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_3 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_3 - \mathbf{p}_1) & \mathbf{z}_2 \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{0} \end{bmatrix}$$

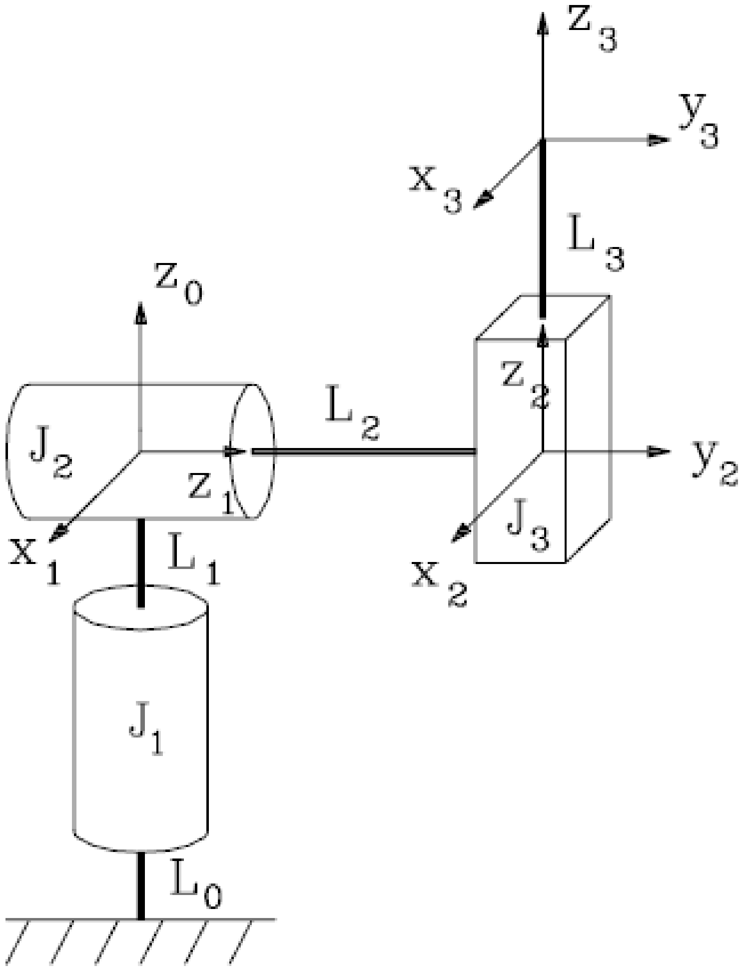
with

$$\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{z}_1 = \begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \quad \mathbf{z}_2 = \begin{bmatrix} C_1 S_2 \\ S_1 S_2 \\ C_2 \end{bmatrix}$$

and

$$\mathbf{p}_0 = \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} -d_2 S_1 \\ d_2 C_1 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} -d_2 S_1 + d_3 C_1 S_2 \\ d_2 C_1 + d_3 S_1 S_2 \\ C_2 d_3 \end{bmatrix}$$

GJ-Example: 3 DOF spherical manipulator



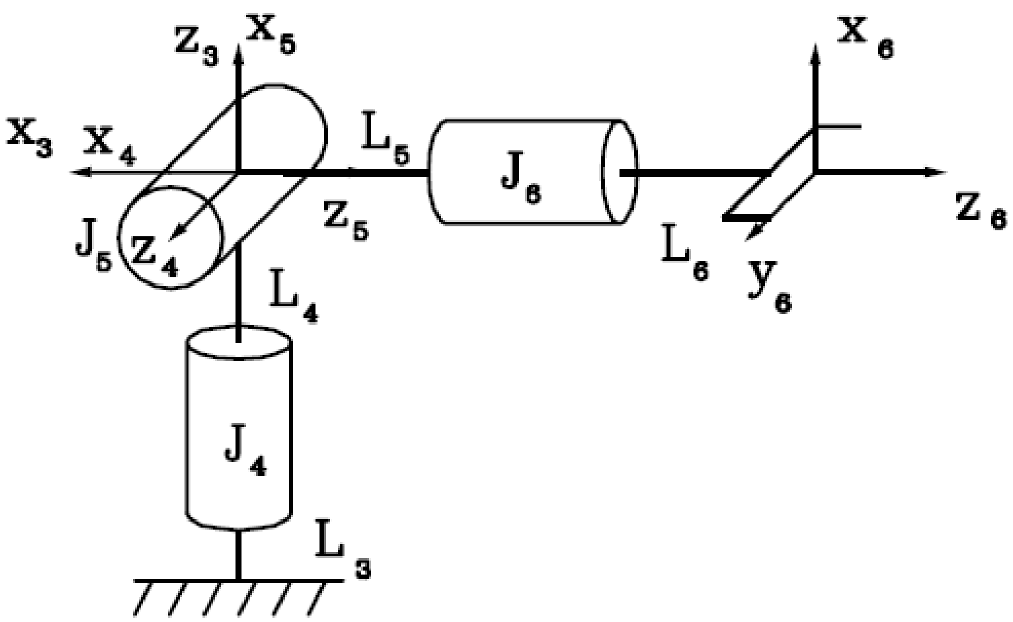
Then

$$J = \begin{bmatrix} -d_2C_1 - d_3S_1S_2 & d_3C_1C_2 & C_1S_2 \\ -d_2S_1 + d_3C_1S_2 & d_3S_1C_2 & S_1S_2 \\ 0 & -d_3S_2 & C_2 \\ 0 & -S_1 & 0 \\ 0 & C_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Note that:

- $\dot{q}_3$ does not affect ω;
- ω_z depends only on $\dot{q}_1$;
- $S_1\omega_y = -C_1\omega_x$.

GJ-Example: 3 DOF spherical wrist



$$J = \begin{bmatrix} -d_6 S_4 S_5 & d_6 C_4 C_5 & 0 \\ d_6 C_4 S_5 & d_6 C_5 S_4 & 0 \\ 0 & -d_6 S_5 & 0 \\ 0 & -S_4 & C_4 S_5 \\ 0 & C_4 & S_4 S_5 \\ 1 & 0 & C_5 \end{bmatrix}$$

By choosing $d_6 = 0$, i.e. the origin of $\mathcal{F}_6$ is in the intersection point of the three joint axes, then

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -S_4 & C_4 S_5 \\ 0 & C_4 & S_4 S_5 \\ 1 & 0 & C_5 \end{bmatrix}$$

With this expression, however, the linear velocity of the end-effector is not computed.

$det(\mathbf{J}) = 0 \implies S_5 = 0$, i.e. $\theta_5 = 0, \pi$.
In this case it is not possible to determine individually $\dot{\theta}_4$ and $\dot{\theta}_6$.

GJ-Example: PUMA 560

Only revolute joints are present:

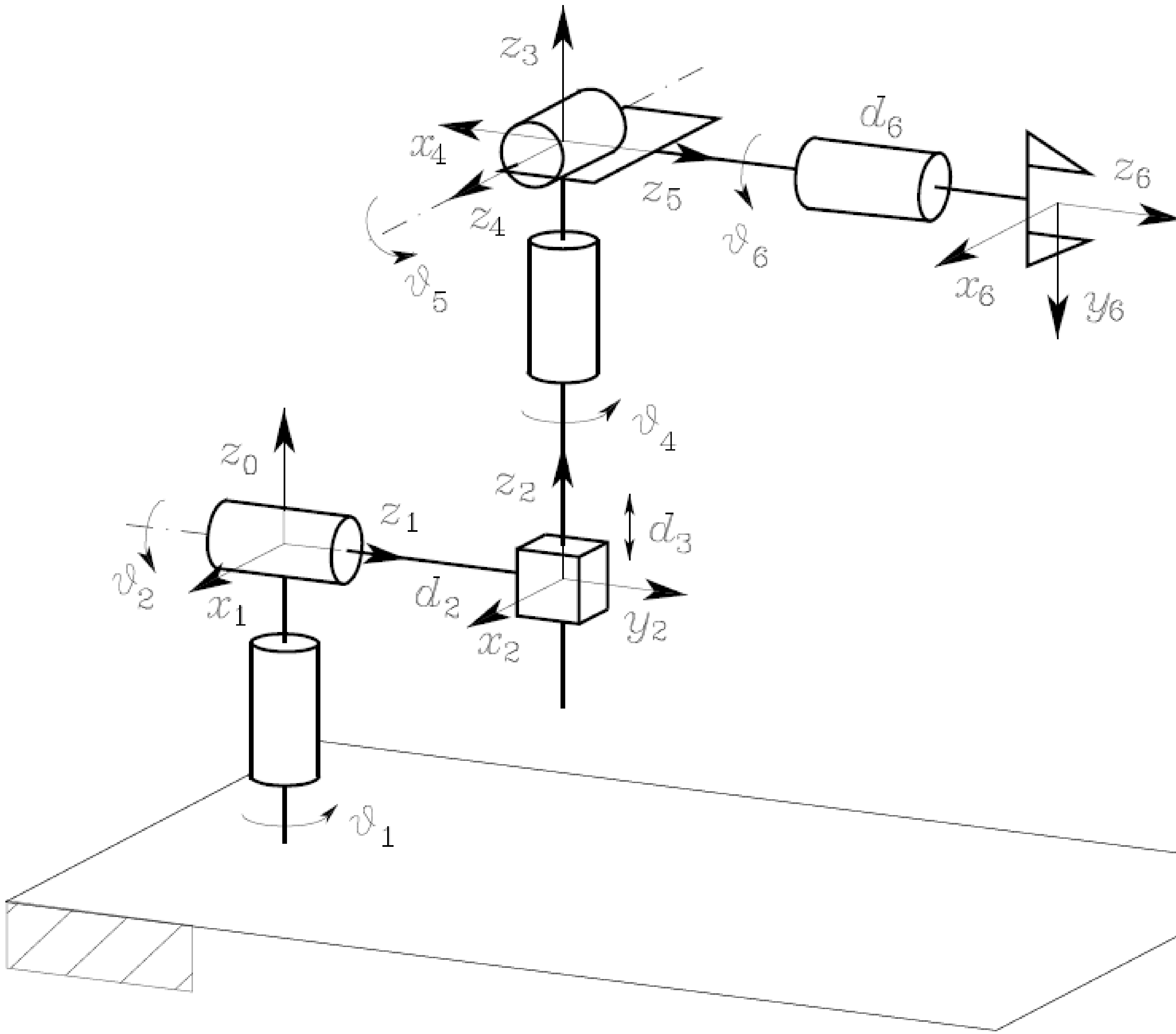
$$\mathbf{J} = \begin{bmatrix} \mathbf{z}_0 \times (\mathbf{p}_6 - \mathbf{p}_0) & \mathbf{z}_1 \times (\mathbf{p}_6 - \mathbf{p}_1) & \mathbf{z}_2 \times (\mathbf{p}_6 - \mathbf{p}_2) & \mathbf{z}_3 \times (\mathbf{p}_6 - \mathbf{p}_3) & \mathbf{z}_4 \times (\mathbf{p}_6 - \mathbf{p}_4) & \mathbf{z}_5 \times (\mathbf{p}_6 - \mathbf{p}_5) \\ \mathbf{z}_0 & \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 & \mathbf{z}_4 & \mathbf{z}_5 \end{bmatrix}$$

GJ-Example: PUMA 560

If $d_6 = 0$:

$$
\mathbf{J} = \left[\begin{array}{cccccc}
-d_3C_1 - S_1(a_2C_2 + d_4S_{23}) & C_1(d_4C_{23} - a_2S_2) & d_4C_1C_{23} & 0 & 0 & 0 \\
-d_3S_1 + C_1(a_2C_2 + d_4S_{23}) & S_1(d_4C_{23} - a_2S_2) & d_4S_1C_{23} & 0 & 0 & 0 \\
0 & a_2C_2 + d_4S_{23} & d_4S_{23} & 0 & 0 & 0 \\
0 & S_1 & S_1 & C_1S_{23} & S_1C_4 - C_1C_{23}S_4 & C_1S_{23}C_5 + C_1C_{23}C_4S_5 + S_1S_4S_5 \\
0 & -C_1 & -C_1 & S_1S_{23} & -C_1C_4 - S_1C_{23}S_4 & S_1S_{23}C_5 + S_1C_{23}C_4S_5 - C_1S_4S_5 \\
1 & 0 & 0 & -C_{23} & -S_{23}S_4 & -C_{23}C_5 + S_{23}C_4S_5
\end{array}\right]
$$

GJ-Example: Stanford manipulator



GJ-Example: Stanford manipulator

$$J = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 & \boldsymbol{z}_3 \times (\boldsymbol{p} - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p} - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p} - \boldsymbol{p}_5) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_3 = \boldsymbol{p}_4 = \boldsymbol{p}_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \boldsymbol{z}_2 = \boldsymbol{z}_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$\boldsymbol{z}_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad \boldsymbol{z}_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$