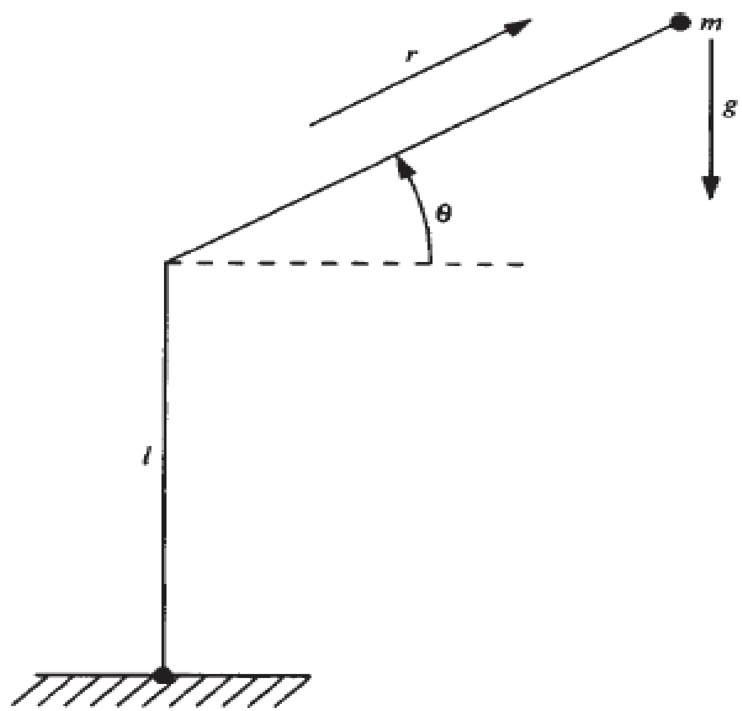
EXAMPLE 3.2–1: Dynamics of a Two-Link Polar Arm

The kinematics for a two-link planar revolute/prismatic (RP) arm are given in Example A.2–3. To determine its dynamics examine Figure 3.2.3, where the joint-variable and joint-velocity vectors are

$$q = \begin{bmatrix} \theta \\ r \end{bmatrix}, \dot{q} = \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}. \tag{1}$$

The corresponding generalized force vector is

$$\tau = \begin{bmatrix} n \\ f \end{bmatrix} \tag{2}$$



a. Kinetic and Potential Energy

The total kinetic energy due to the angular motion $\dot{\boldsymbol{\theta}}$ and the linear motion $\dot{\boldsymbol{r}}$ is

$$K = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 \tag{3}$$

and the potential energy is

$$P = mgr\sin\theta \tag{4}$$

b. Lagrange's Equation

The Lagrangian is

$$L = K - P = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{r}^2 - mgr\sin\theta \tag{5}$$

Now we obtain

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{\theta}} \\ \frac{\partial L}{\partial \dot{r}} \end{bmatrix} = \begin{bmatrix} mr^2\dot{\theta} \\ m\dot{r} \end{bmatrix} \tag{6}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} \\ m\ddot{r} \end{bmatrix} \tag{7}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} -mgr\cos\theta \\ mr\dot{\theta}^2 - mg\sin\theta \end{bmatrix} \tag{8}$$

Therefore, (3.2.14) shows that the arm dynamical equations are

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos\theta = n \tag{9}$$

$$m\ddot{r} - mr\theta^2 + mg\sin\theta = f \tag{10}$$

c. Manipulator Dynamics

By using vectors, the arm equations may be written in a convenient form. Indeed, note that

$$\begin{bmatrix} mr^2 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + \begin{bmatrix} 2mr\dot{r}\dot{\theta} \\ -mr\dot{\theta}^2 \end{bmatrix} + \begin{bmatrix} mgr\cos\theta \\ mg\sin\theta \end{bmatrix} = \begin{bmatrix} n \\ f \end{bmatrix} \tag{11}$$

We symbolize this vector equation as

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau \tag{12}$$

Note that, indeed, the inertia matrix *M(q)* is a function of *q* (i.e., of θ and *r*), the *Coriolis/centripetal vector* $V(q,\dot{\boldsymbol{q}})$ is a function of *q* and $\dot{\boldsymbol{q}}$, and the *gravity vector G(q)* is a function of *q*.

EXAMPLE 3.2–2: Dynamics of a Two-Link Planar Elbow Arm

In Example A.2–2 are given the kinematics for a two-link planar RR arm. To determine its dynamics, examine Figure 3.2.4, where we have assumed that the link masses are concentrated at the ends of the links. The joint variable is

$$q=[\theta_1 \ \ \theta_2]^T \tag{1}$$

and the generalized force vector is

(2)

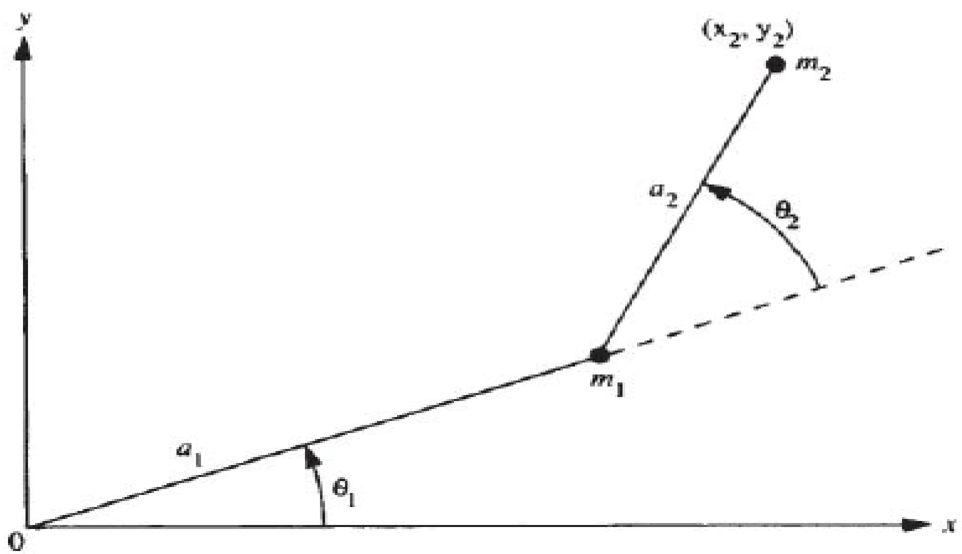


Figure 3.2.4: Two-link planar RR arm.

with τ_1, and τ_2 torques supplied by the actuators.

a. Kinetic and Potential Energy

For link 1 the kinetic and potential energies are

$$K_1 = \tfrac{1}{2} m_1 a_1^2 \dot{\theta}_1^2 \tag{3}$$

$$P_1 = m_1 g a_1 \sin\theta_1 . \tag{4}$$

For link 2 we have

$$x_2 = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \tag{5}$$

$$y_2 = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \tag{6}$$

$$\dot{x}_2 = -a_1 \dot{\theta}_1 \sin\theta_1 - a_2(\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \tag{7}$$

$$\dot{y}_2 = a_1 \dot{\theta}_1 \cos\theta_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2), \tag{8}$$

so that the velocity squared is

$$\begin{aligned} v_2^2 &= \dot{x}_2^2 + \dot{y}_2^2 \\ &= a_1^2 \dot{\theta}_1^2 + a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos\theta_2 . \end{aligned} \tag{9}$$

Therefore, the kinetic energy for link 2 is

$$\begin{aligned} K_2 &= \tfrac{1}{2} m_2 v_2^2 \\ &= \tfrac{1}{2} m_2 a_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ &\quad + m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos\theta_2 . \end{aligned} \tag{10}$$

The potential energy for link 2 is

$$P_2 = m_2 g y_2 = m_2 g[a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2)]. \tag{11}$$

b. Lagrange's Equation

The Lagrangian for the entire arm is

$$\begin{aligned} L = & K - P = K_1 + K_2 - P_1 - P_2 \\ & = \tfrac{1}{2}(m_1 + m_2) a_1^2 \theta_1^2 + \tfrac{1}{2} m_2 a_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ & \quad + m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2) \cos \theta_2 \\ & \quad - (m_1 + m_2) g a_1 \sin \theta_1 \\ & \quad - m_2 g a_2 \sin(\theta_1 + \theta_2). \end{aligned} \tag{12}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} =&(m_1 + m_2)a_1^2\dot{\theta}_1 + m_2 a_2^2(\dot{\theta}_1 + \dot{\theta}_2) \\
&+ m_2 a_1 a_2 (2\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} =&(m_1 + m_2)a_1^2\ddot{\theta}_1 \\
&+ m_2 a_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 (2\ddot{\theta}_1 + \theta_2)\cos\theta_2 \\
&- m_2 a_1 a_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 \\
\frac{\partial L}{\partial \theta_1} =& - (m_1 + m_2) g a_1 \cos\theta_1 - m_2 g a_2 \cos(\theta_1 + \theta_2) \\
\frac{\partial L}{\partial \dot{\theta}_2} =&m_2 a_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2 a_1 a_2 \dot{\theta}_1 \cos\theta_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} =&m_2 a_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 a_1 a_2 \ddot{\theta}_1 \cos\theta_2 \\
&- m_2 a_1 a_2 \dot{\theta}_1\dot{\theta}_2 \sin\theta_2 \\
\frac{\partial L}{\partial \theta_2} =& - m_2 a_1 a_2 (\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)\sin\theta_2 - m_2 g a_2 \cos(\theta_1 + \theta_2).
\end{aligned}
$$

Finally, according to Lagrange's equation, the arm dynamics are given by the two coupled nonlinear differential equations

$$\begin{aligned}\tau_1 &= \left[(m_1+m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2\cos\theta_2\right]\ddot{\theta}_1 \\ &\quad + \left[m_2a_2^2 + m_2a_1a_2\cos\theta_2\right]\ddot{\theta}_2 - m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 \\ &\quad + (m_1+m_2)ga_1\cos\theta_1 + m_2ga_2\cos(\theta_1+\theta_2) \end{aligned} \tag{13}$$

$$\begin{aligned}\tau_2 &= \left[m_2a_2^2 + m_2a_1a_2\cos\theta_2\right]\ddot{\theta}_1 + m_2a_2^2\ddot{\theta}_2 + m_2a_1a_2\dot{\theta}_1^2\sin\theta_2 \\ &\quad + m_2ga_2\cos(\theta_1+\theta_2). \end{aligned} \tag{14}$$

$$\begin{aligned} & M(q)\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} -m_2a_1a_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)\sin\theta_2 \\ m_2a_1a_2\dot{\theta}_1^2\sin\theta_2 \end{bmatrix} \\ & + \begin{bmatrix} m_1+m_2)ga_1\cos\theta_1 + m_2ga_2\cos(\theta_1+\theta_2) \\ m_2ga_2\cos(\theta_1+\theta_2) \end{bmatrix} \\ & = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}. \end{aligned} \tag{15}$$

where

$$M(q) =$$
$$\begin{bmatrix} (m_1+m_2)a_1^2 + m_2a_2^2 + 2m_2a_1a_2\cos\theta_2 & m_2a_2^2 + m_2a_1a_2\cos\theta_2 \\ m_2a_2^2 + m_2a_1a_2\cos\theta_2 & m_2a_2^2 \end{bmatrix} \tag{16}$$

These manipulator dynamics are in the standard form

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau \tag{17}$$

with $M(q)$ the inertia matrix, $V(q,\dot{\boldsymbol{q}})$ the Coriolis/centripetal vector, and $G(q)$ the gravity vector. Note that $M(q)$ is symmetric.

Table 3.3.1: The Robot Equation and Its Properties

$$M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q) + \tau_d = \tau$$

or

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau$$

where

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F(\dot{q}) + G(q)$$

Inertia Matrix:

$M(q)$ is symmetric and positive definite.

$\mu_1 I \leq M(q) \leq \mu_2 I$

$m_1 \leq \|M(q)\| \leq m_2$

Coriolis/Centripetal Vector:

$V(q,\dot{q})$ is quadratic in $\dot{q}$

$\|V(q,\dot{q})\| \leq v_b \|\dot{q}\|^2$

$V(q,\dot{q}) = V_m(q,\dot{q})\dot{q}$

$S(q,\dot{q}) \equiv \dot{M}(q) - 2V_m(q,\dot{q})$ is a skew-symmetric matrix

Friction Terms:

$$F(\dot{q}) = F_v(\dot{q}) + F_d(\dot{q})$$
$$F_v = \operatorname{diag}\{v_1\}$$
$$F_d(\dot{q}) = K_d \operatorname{sgn}(\dot{q}), \text{ with } K_d = \operatorname{diag}\{k_i\}$$
$$\|F_v\dot{q} + F_d(\dot{q})\| \le v\|\dot{q}\| + k$$

Gravity Vector:

$$\|G(q)\| \le g_b$$

Disturbance Term:

$$\|\tau_d\| \le d$$

Linearity in the Parameters:

$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q)$$
$$= M(q)\ddot{q} + N(q,\dot{q}) \equiv W(q,\dot{q},\ddot{q})\varphi$$